Limit Cycle, Potential Landscape, and Complex Dynamics

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Abstract

The existence of potential landscape in complex dynamics, though very appealing, has been controversial. Recently one of the present authors proposed a general construction. Because of technical difficulty, such a construction has not been explicitly demonstrated for complicated dynamics, such as chaotic ones. As a step towards such goal, here we demonstrate that the potential function can coexist with limit cycles in nonlinear and dissipative dynamical systems. The potential function indeed drives dynamics and determines the final steady state distribution similar to the usual situation in physics, that is, it has both local and global meanings.

Our demonstration consists of three steps: We first show the existence of limit from a typical physics setting by an explicit construction in two dimensions. When approaching to the limit cycle, the strength of the potential gradient goes to zero; the magnetic field goes to zero in the same order of the potential gradient and changes sign at the limit cycle; and the friction goes to zero at higher order than that of potential gradient. The dynamics at the limit cycle is conserved in this limit. The diffusion matrix is nevertheless finite at the limit cycle. Second, based on such physics knowledge we can construct the potential in the dynamics with limit cycle in a typical dynamical systems setting. Third, we argue that such a construction can be in principle carried out in a general situation combined with on a novel method of dealing with stochastic differential equation. This novel method is different from both Ito and Stratonovich calculus shown explicitly in the present article.

I. INTRODUCTION

Dynamics with limit cycles are abundant in complex systems, whether or not perturbed by noise. They are among the prototype systems in the study of nonlinear systems[1–3]. Until recently, one of the most useful concepts in physics, the potential energy function, had been found to be controversial when applying beyond physical sciences[2–6], though its existence has been emphatically stressed[7]. Hence, it has been concluded that the Lyapunov function or potential function with periodic attractors could not exist, for example, in the neural network computing[5]. Similar statements can also be found in many other fields such as in evolutionary biology[6,8]. One of the present authors has proposed a method to construct the potential function in complex dynamics[4] and the dynamics dominated by fixed points has been investigated in detailed[9]. However, for the more complicated dynamics the construction remains to be explicitly implemented. The purpose of the present article is to give an explicit demonstration that at least for the case of limit cycle dynamics the potential function can be constructed in an explicit manner. In this way we provides a critical link to more complicated dynamics.

Our construction is based on a novel way of handling the standard stochastic differential equation[2,4]:

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, t) + \zeta(\mathbf{q}, t) \ . \tag{1}$$

Here $\mathbf{f}(\mathbf{q}, t)$ is the deterministic nonlinear drive of the system and the stochastic drive is $\zeta(\mathbf{q}, t)$. $\mathbf{f}$ and $\mathbf{q}$ are vector in n dimensional space. For simplicity we will assume that $\mathbf{f}$ is a smooth function whenever needed. The stochastic drive $\zeta(\mathbf{q}, t)$ in Eq.(1) is assumed to be Gaussian and white with zero mean and the variance,

$$\langle \zeta(\mathbf{q}, t)\zeta^{\tau}(\mathbf{q}, t') \rangle = 2D(\mathbf{q}) \, \epsilon \, \delta(t - t') \ , \tag{2}$$

and zero mean, $\langle \zeta(\mathbf{q}, t) \rangle = 0$. The $n \times n$ matrix D is the diffusion matrix. The question is on the construction of the potential function out of Eq.(1), with both local and global meanings.

The novel procedure can be understood as the zero mass limit of a generalized Klein-Kramers equation[4]. In this limit the steady state distribution can be established from the Klein-Kramers equation without the dilemma raised from Ito vs Stratonovich calculus. Therefore it is not surprising that a different perspective can be obtained other than those in Ref.[2]. We believe that our following demonstration provides an important step for an alternative understanding of the dynamical structure in complex systems.

We will demonstrate the co-existence of the potential function with limit cycle in three steps in three sections, starting from the simple and concrete construction in two dimensions to the general and existence demonstration. Because of the technical difficulty in the problem, we will present results in an elementary manner to help establishing the local and global properties of a potential function. In fact, almost all of our results are reached algebraically. In section II we explicitly demonstrate how to construct a limit cycle based on our physics knowledge, with the potential. This is different from the usual approach in dynamical systems. In section III we demonstrate how to construct a potential function in the presence of a limit cycle in the dynamical systems setting. In section IV we outline a general construction of potential function in a broad class of dynamical systems, including limit cycles. In section V we discuss two mathematical subtleties and a previous approach. In section VI we conclude that the potential function can co-exist with limit cycle and with possibly more complicated dynamics.

II. LIMIT CYCLE: PHYSICS' POINT OF VIEW

The goal of this section is twofold: to provide a concrete starting point for the rest of the paper and to obtain an explicit example of potential function with expected properties in the presence of limit cycle.

In physics, the general dynamical equation for a *massless* particle in two dimensional state space may be expressed as, with both deterministic and stochastic forces[10]:

$$[S(\mathbf{q}) + T(\mathbf{q})]\dot{\mathbf{q}} = -\nabla\psi(\mathbf{q}) + \xi(\mathbf{q}, t) \ , \tag{3}$$

and supplemented by the relationship on the stochastic force:

$$\langle \xi(\mathbf{q}, t)\xi^{\tau}(\mathbf{q}, t') \rangle = 2S(\mathbf{q}) \, \epsilon \, \delta(t - t') \ , \tag{4}$$

and $\langle \xi(\mathbf{q}, t) \rangle = 0$. Here $\mathbf{q}^{\tau} = (q_1, q_2)$ with q_1, q_2 the two Cartesian coordinates of the state space, which may be perceived as the position space of the massless particle. The transpose is denoted by the superscript τ, and $\dot{\mathbf{q}} = d\mathbf{q}/dt$.

The scalar function ψ corresponds to the usual potential energy in physical sciences. Its graphical representation in the state space is a landscape. The antisymmetric matrix T

represents the dynamics which conserves the potential, corresponding to the Lorentz force in physics, determined by the magnetic field. The matrix S represents the dynamics which decreases the potential, the dissipation in physics. This matrix may be called the friction matrix. The simplest realization of the friction matrix is a constant matrix

$$S = \eta(\mathbf{q}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with the friction coefficient η. The realization of the antisymmetric matrix is

$$T = B(\mathbf{q}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with B as the strength of magnetic field. With such a realization, the left hand side of Eq.(3) becomes

$$\eta(\mathbf{q})\dot{\mathbf{q}} + B(\mathbf{q})\hat{z} \times \dot{\mathbf{q}} \ ,$$

the very familiar form of frictional and Lorentz forces. Here $\hat{z}$ is the unit vector perpendicular to the state space formed a plane by q_1 and q_2, indicating the direction of the magnetic field B with the electric charge taken to be 1.

The friction matrix is connected to the stochastic force ξ by Eq.(4), which guarantees that it is semi-positive definite and symmetric. All T, S, ψ can be nonlinear functions of the state variable $\mathbf{q}$ as well as the time t. The numerical parameter ϵ corresponds to an effective temperature, which can be taken to be zero to recover the deterministic dynamics. It has been shown that if a steady state distribution $\rho(\mathbf{q})$ in state space exists,

$$\rho(\mathbf{q}) = \frac{1}{Z} \exp\left(- \frac{\psi(\mathbf{q})}{\epsilon} \right) \ , \tag{5}$$

a Boltzmann-Gibbs type distribution function[4]. Here Z is the partition function $Z = \int d\mathbf{q} \ \exp(-\psi(\mathbf{q})/\epsilon)$. Eq.(5) implies that for dynamics which repeats itself indefinitely, such as limit cycle, the potential should be the same along such trajectory. Thus, the local meaning of the potential function is manifested in Eq.(5) and the global meaning in Eq.(5). Though Eq.(5) indeed confirms that the potential function is the ultimate goal of the dynamics, at any given time, it is only part of the dynamics process. Other three, the friction matrix S, the transverse matrix T, and the noise ξ, are needed, as expressed by Eq.(3).

It should be pointed that by construction the potential function $\psi(\mathbf{q})$ always exists. This is one of most useful and important quantitative concepts in physics. If the stochastic force could be set to be zero, $\xi(\mathbf{q}, t) = 0$, that is, the deterministic dynamics, the dynamics of this massless particle always decreases its potential energy:

$$\begin{aligned} \dot{\mathbf{q}}^\tau \nabla \psi(\mathbf{q}) &= -\dot{\mathbf{q}}^\tau [S(\mathbf{q}) + T(\mathbf{q})] \dot{\mathbf{q}} \\ &= -\dot{\mathbf{q}}^\tau S(\mathbf{q}) \dot{\mathbf{q}} \\ &\leq 0 \, . \end{aligned} \tag{6}$$

The zero occurs only at the invariant sets: fixed points (point attractors), limit cycles (periodic attractors) and/or more complicated ones. Hence the potential function has the usual meaning of Lyapunov function.

To explicitly model a limit cycle, we choose following forms for the friction matrix S, the anti-symmetric matrix T, and the potential function ψ, assuming the limit cycle occurs at $q = \sqrt{q_1^2 + q_2^2} = 1$:

$$S = \frac{(q^2-1)^2}{(q^2-1)^2+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{7}$$

$$T = (q-1)\frac{q^2}{(q^2-1)^2+1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{8}$$

$$\psi = \frac{1}{2}(q-1)^2 \, . \tag{9}$$

The potential function ψ given in Eq.(9) is rotational symmetric in the state space and $|\nabla\psi| = |q-1|$. It has a local maximum $\psi = 1/2$ at $q = 0$, which is a cusp, and the minimum $\psi = 0$ at $q = 1$, which is a cycle in the state space. Hence the potential function takes the shape of a Mexican hat.

Evidently, if the friction matrix S would be zero, the dynamical trajectory of the massless particle would move along the equal potential contour determined by the initial condition, which would be a cycle according to above chosen potential. In the presence of nonzero friction matrix, this is not true. What will be our concern is the behavior near the minimum of the potential function: When q is sufficiently close to 1, does the particle trajectory asymptotically approach the cycle of $q = 1$ and eventually coincide with it? If the answer is positive, we have a limit cycle dynamics. We will demonstrate below that it is indeed possible.

For a deterministic dynamics, we can set $\epsilon = 0$ in Eq.(3) and (4): setting the stochastic force to be zero. The dynamical equation can then be rewritten as

$$\dot{\mathbf{q}} = -[S(\mathbf{q}) + T(\mathbf{q})]^{-1} \nabla \psi(\mathbf{q}, t) \ . \tag{10}$$

With the choice of Eqs. (7-9), we have

$$\begin{aligned} [S+T]^{-1} &= \frac{1}{\det(S+T)} \left[\frac{(q^2-1)^2}{(q^2-1)^2+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right. \\ & \left. -(q-1)\frac{q^2}{(q^2-1)^2+1}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \end{aligned} \tag{11}$$

and $\det(S+T) = [(q^2-1)^2/((q^2-1)^2+1)]^2 + [(q-1)^2 q^4/[(q^2-1)^2+1]^2$. Near $q = 1$, we have

$$\begin{aligned} [S+T]^{-1} &= \frac{1}{q-1} \left[-(1-2(q-1)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right. \\ & \left. + 4(q-1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + O((q-1)^2) \right] \ . \end{aligned} \tag{12}$$

In terms of radial coordinate q and azimuthal angle θ in the polar coordinate representation of the state space, using the small parameter expansion given in Eq.(12) following Eq.(10) we have, to the order of $q-1$,

$$\dot{q} = -4(q-1) \ , \tag{13}$$

$$\dot{\theta} = 1 - 2\frac{q-1}{q} \ . \tag{14}$$

The solution is

$$q(t) = 1 + \delta q_0 \exp\{-4t\} \ , \tag{15}$$

$$\begin{aligned} \theta(t) &= \theta_0 + t + \frac{1}{4} \ln \frac{1 + \delta q_0 \exp\{-4t\}}{1 + \delta q_0} \\ & + \frac{1}{2} \ln\left(1 + \delta q_0 \exp\{-4t\}\right) \ . \end{aligned} \tag{16}$$

Here δq_0 ($|\delta q_0| << 1$) is the starting radial position of the particle measured from $q = 1$ and θ_0 its the starting azimuth angle. Indeed, the solution demonstrates the asymptotical approaching to the cycle $q = 1$, and the motion never stops. Unstable and metastable limit cycles may be constructed in the similar manner.

Though the above construction does show that based on the physics knowledge one can construct limit cycle with the potential, the example of Eqs.(7-9) appears contrived. We should, however, point out that there are several generic features in our construction.

(a) To have an indefinitely motion on a closed trajectory, because of energy conservation, the friction, or better the friction matrix here, must be zero.

(b) Because of the asymptotic motion is on the equal potential contour, a Lorentz force type must exist to keep the motion on the contour. This means that the antisymmetric matrix should be finite along the equal motion contour when the potential gradient is finite. The speed of the massless particle moving along the contour will be determined by the ratio of the strength of the Lorentz like force to that of the gradient of the potential.

(c) The limit cycle should be robust: Small parameter changes should only have a small effect on the limit cycle, the (stable) limit cycle should be at the minimum of the potential. As a consequence, the potential gradient at the minimum is zero, which would imply that the friction matrix must go to zero faster than that of the potential gradient when approaching to the minimum to avoid the potential function taking singular values. This means that at the limit cycle the dynamics is conserved.

(d) Furthermore, to ensure the massless particle moves in the same direction on both sides of the limit cycle, the magnetic field should change its sign at the limit cycle. All those features are explicitly implemented in the choice, Eqs.(7-9).

Three additional remarks are in order.

(e) For simplicity of calculation we have chosen the friction matrix to be a constant matrix in Eq.(7). One can check that any positive definite symmetric matrix can lead to above same conclusion, as long as its strength goes to zero in a higher order comparing to that of the potential gradient.

(f) Although the friction matrix S is zero when approaching to the limit cycle, $[S+T]^{-1}+[S-T]^{-1}$ is not, on which we will come back later when discussing the diffusion matrix D in section V.

(g) It should be emphasized here that the potential function has a dual role: Its gradient is the driving force in dynamics, expressed in Eq.(3) or Eq.(10), and it determines the final steady state distribution, expressed in Eq.(5).

III. LIMIT CYCLE: DYNAMICAL SYSTEMS' POINT OF VIEW

The section II we show that starting with a potential function one can construct limit cycle. In this section we will shown the reverse. The demonstration of the co-existence of limit cycle with the potential function in section II may appear special: The massless particle moves along the potential minimum with both zero friction and zero transverse matrices, a nice picture from physics. A natural question is whether or not in a typical limit cycle in dynamical systems a potential function can be constructed. We will also demonstrate in this section that the answer to such question is a straightforward yes.

It has been suggested that a simple limit cycle in two dimensions would take following form for its dynamical equation in a polar coordinate representation[11]:

$$\dot{q} \;=\; R(q) \; , \tag{17}$$

$$\dot{\theta} \;=\; \Phi(q) \; . \tag{18}$$

Here the smooth functions R, Φ have properties $R(q=1)=0$ is a fixed point in the radial coordinate and $\Phi(q=1) = constant$. We point out that mathematically any shape of limit cycle in two dimension can be deformed into a cycle and that near this limit cycle the dynamical equation can be mapped into above form by a nonlinear coordinate transformation. Hence Eq.(17) and (18) may be regarded as a representation for a generic limit cycle in two dimensions.

The comparison between Eqs.(13,14) and Eqs.(17,18) immediately suggests what considered in section II is just such a typical limit cycle. The construction of potential function from Eq.(17) and (18) is also immediately suggested: Given functions $R(q)$ and $\Phi(q)$, there exist the friction matrix S, the potential function ψ, and the antisymmetric matrix T. In fact, we have three independent quantities to be constructed instead of two. We will come back to this uniqueness question in following two sections. We may conclude here that the potential function and a limit cycle coexist for such cases.

To explicitly construct the potential function from Eq.(17) and (18), we go back to Eq.(10). Dropping the time-dependence in those functions and defining $[S+T]^{-1} = D + Q$, Eq.(10) may be rewritten as

$$\dot{\mathbf{q}} = -[D(\mathbf{q}) + Q(\mathbf{q})]\nabla\psi(\mathbf{q}) \; . \tag{19}$$

Here D is a symmetric and positive definite matrix and Q an antisymmetric matrix. For simplicity, we choose D to be the identify matrix,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

and

$$Q = b(q) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Here $b(q)$ is a scale function.

In accordance with the rotational symmetry in Eq.(17) and (18), we also choose the potential function depending only on the radial coordinate q, $\psi(q)$. With above choices, in the polar coordinates Eq.(19) implies

$$\begin{aligned} \dot{q} &= -\frac{d\psi(q)}{dq} , && (20) \\ q\dot{\theta} &= -b(q)\frac{d\psi(q)}{dq} . && (21) \end{aligned}$$

Comparing above two equations with Eq.(17) and (18) we have

$$\begin{aligned} \frac{d\psi(q)}{dq} &= -R(q) , && (22) \\ b(q) &= q\frac{\Psi(q)}{R(q)} . && (23) \end{aligned}$$

Thus,

$$\begin{aligned} S(q) + T(q) &= [D(q) + Q(q)]^{-1} \\ &= \begin{pmatrix} 1 & q\frac{\Psi(q)}{R(q)} \\ -q\frac{\Psi(q)}{R(q)} & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{1 + \left(q\frac{\Psi(q)}{R(q)}\right)^2} \begin{pmatrix} 1 & -q\frac{\Psi(q)}{R(q)} \\ q\frac{\Psi(q)}{R(q)} & 1 \end{pmatrix} . && (24) \end{aligned}$$

This gives,

$$\begin{aligned} S(q) &= \frac{1}{1 + \left(q\frac{\Psi(q)}{R(q)}\right)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , && (25) \\ T(q) &= \frac{1}{1 + \left(q\frac{\Psi(q)}{R(q)}\right)^2}\, q\frac{\Psi(q)}{R(q)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . && (26) \end{aligned}$$

Eqs.(22,25,26) are one explicit construction of potential function for the limit cycle dynamics described by Eq.(17) and (18), corresponding to the dynamics described by Eq.(3) and (4).

When approaching to the limit cycle, $q \to 1$, $R(q) = O(q-1)$ and $\Phi(q) = O(1)$, and,

$$
\begin{aligned}
\frac{d\psi(q)}{dq} &= O(q-1) \ , && (27) \\
T(q) &= O(q-1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , && (28) \\
S(q) &= O((q-1)^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ . && (29)
\end{aligned}
$$

Clearly, this has the same structures as those in Eqs.(7-9). The friction matrix S indeed vanishes faster, though the diffusion matrix D here always remains a constant. The dynamics eventually becomes non-dissipative along the limit cycle from the point of view of Eq.(3). This completes the discussion of construction of potential function for a limit cycle in two dimensions.

Nevertheless, we should point out an interesting feature. On one side it is known from the theory of dynamical systems that a limit cycle is robust[3]. Any small parameter change in the equation would not lead to its disappearing. On other side, from the demonstration in section II which is based on the understanding from physics, the existence of the limit cycle is a result of a very delicate balance between all dynamical elements: The friction matrix, the gradient of potential function, and the antisymmetric matrix. They are all zero at the limit cycle, and when approaching to the limit cycle, the friction matrix vanishes faster. How this paradoxical feature would play a role in our better understanding limit cycle and its control will be an interesting problem for further exploration.

IV. GENERAL CONSTRUCTION FROM DYNAMICAL SYSTEMS

A particular challenging and careful reader would observe that we have not given the explicit construction of potential function for a limit cycle in arbitrary dimensions. One would wonder that there might exist limit cycles which will not be ones as discussed above such that the potential function might not exist. We have indeed been unable to give an explicit construction for a generic limit cycle (a periodic attractor). However, we point out that a general construction has been done for a generic fixed point (a point attractor)[9].

Furthermore, we do have a construction of potential function for a generic situation. Such a construction suggests that it is possible to do so generally for arbitrary limit cycles. Therefore the existence of potential function can be expected when combining this method with the observations in section II. The general method has been discussed elsewhere[4]. Here we outline the key ideas to make the present demonstration complete.

We restore the stochastic force back into the general equation in dynamical systems and consider an arbitrary dimension n ($n \geq 2$), same as Eq.(1):

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}) + \zeta(\mathbf{q}, t) \ .$$

Here $\mathbf{f}(\mathbf{q})$ is the deterministic nonlinear drive of the system and the stochastic drive is $\zeta(\mathbf{q}, t)$, which differs from that in Eq.(1) but comes from the same dynamical source. For simplicity we will assume that $\mathbf{f}$ is a smooth function whenever needed. To be consistent with Eq.(1) and (2) when extended to dimension $n \geq 2$, the stochastic drive in Eq.(1) is also assumed to be Gaussian and white with the variance, same as Eq.(2),

$$\langle \zeta(\mathbf{q}, t) \zeta^{\tau}(\mathbf{q}, t') \rangle = 2D(\mathbf{q}) \, \epsilon \, \delta(t - t') \ ,$$

and zero mean, $\langle \zeta(\mathbf{q}, t) \rangle = 0$.

Assuming that both Eq.(1) and (3) describe the same dynamics. It is rather easy to derive Eq.(3) from Eq.(1). The difficult part is to derive Eq.(1) from Eq.(3). Using Eq.(3) to eliminate $\dot{\mathbf{q}}$ in Eq.(1), and noticing that the dynamics of noise and the state variable behave independently, we have, the deterministic part,

$$[S(\mathbf{q}) + T(\mathbf{q})]\mathbf{f}(\mathbf{q}) = -\nabla \psi(\mathbf{q}) \ , \tag{30}$$

and the stochastic part,

$$[S(\mathbf{q}) + T(\mathbf{q})]\zeta(\mathbf{q}, t) = \xi(\mathbf{q}, t) \ . \tag{31}$$

Those two equations suggest a rotation in state space.

Multiplying Eq.(31) by its own transpose of each side and carrying out the average over stochastic drive, we have

$$[S(\mathbf{q}) + T(\mathbf{q})]D(\mathbf{q})[S(\mathbf{q}) - T(\mathbf{q})] = S(\mathbf{q}) \ . \tag{32}$$

In obtaining Eq.(32) we have also used Eq.(2) and (4). Eq.(32) gives $n(n+1)/2$ conditions because it is symmetric under the transpose operation.

Using the property of the potential function ψ: $\nabla \times \nabla\psi = 0$ $[(\nabla \times \nabla\psi)_{ij} = (\nabla_i\nabla_j - \nabla_i\nabla_j)\psi\ ]$, Eq.(30) leads to

$$\nabla \times [[S(\mathbf{q}) + T(\mathbf{q})]\mathbf{f}(\mathbf{q})] = 0 \ , \tag{33}$$

which gives $n(n-1)/2$ conditions because the antisymmetric condition implied.

Combining both Eq.(32) and (33), we have total n^2 conditions, which is exactly the number to determine the $n \times n$ matrix $[S(\mathbf{q}) + T(\mathbf{q})]$, when supplemented by appropriate boundary condition needed by the partial differential equations of Eq.(33). This additional boundary condition should be the delicate balance condition discussed in section II and III: When approaching the limit cycle, the friction matrix must approach to zero faster than the potential gradient as well as the antisymmetric matrix. Once $[S + T]$ is known, it is straightforward to obtain the potential function according to Eq.(30), hence construct Eq.(3) and (4) from Eq.(1) and (2).

The general existence of the potential function implies that indeed the energy landscape concept in physics should be one of the most important concepts in dynamical systems, too. Thus, not only the potential function exist in periodic attractors, may also in more complicated ones. This would have implications in other fields where dynamical behaviors are the primary concerns. For example, we have shown elsewhere[12] that the mathematical structure of population genetics can be formulated in such a way to incorporate the important concept of potential function.

V. DISCUSSIONS

A. mathematical subtleties

There are two mathematical subtleties. First, it is known that even with a limit cycle the dynamics is dissipative, reflecting by the fact that in general $\nabla \cdot \mathbf{f} \neq 0$, where $\mathbf{f}$ is defined by Eq.(1). This is also expressed by the fact that the so-called diffusion matrix, D defined in Eq.(2), is finite even at the limit cycle. Because it is dissipative, it would be difficult to conceive a constant potential function (or a Lyapunov function) along the limit cycle on which the dynamics repeats itself indefinitely. The delicate point is that, as shown in our above demonstration, that the friction matrix S is zero along the limit cycle does not implies the diffusion matrix D is zero. In fact, it is finite according to Eq.(32):

$$D = \frac{1}{2}[(S+T)^{-1} + (S-T)^{-1}] \ . \qquad (34)$$

An explicit verification can be obtained from Eq.(12) or (24). An interesting and direct implication of present construction is that the statement, one cannot construct Hopfield potential (or Lyapunov function) in limit cycle in computing by attractors in the neural networks[5], is not true.

We come to the second subtlety. For deterministic dynamics, if one finds one Lyaponov function, one finds many. This is illustrated by the present construction that additional information from the noise is needed to make the construction unique: different diffusion matrix would lead to different potential function. We already encountered this issue when construction potential function from Eq.(17) and (18), without the specification of noise. However, this freedom may provide a method to select the best suitable Lyapunov function or potential function to ones own problem by choosing appropriate form of the diffusion matrix.

It is worthwhile to point out that the gradient systems in dynamical systems theory corresponds to the zero transverse matrix, $T = 0$, in the present construction. No limit cycle is possible in this case. In dynamics described by gradient systems the trajectory could follow the most rapid descendant path along the landscape defined by the potential. In this case it is easy to identify the potential function as the landscape function. For a general dynamics where the transverse matrix is not zero, the trajectory would not follow the most rapid descendant route along the potential, as expressed by Eq.(3) or Eq.(10). Nevertheless, the meaning of the potential function remains the same as that in gradient systems: driving the dynamics and determining the final steady state distribution.

B. historical perspective

We should also point out that the Mexican hat type potential function has been suggested for limit cycle long ago[13]. This idea has been further developed by Rytov[14] and by Lax[15], among others, and may be explicitly summarized in the following form[16]:

$$\dot{\mathbf{q}} = -[D(\mathbf{q}) + Q(\mathbf{q})]\nabla V(\mathbf{q}) + \zeta(\mathbf{q}, t) \ . \qquad (35)$$

The connection between the noise ζ and the diffusion matrix is the same as in Eq.(2). With appropriately choosing scalar function V of the Mexican hat type and the antisymmetric matrix Q, it was noticed that limit cycle dynamics can be generated[14–16]. With either Ito or Stratonovich type treatment of stochastic integration, two of the most popular treatments of stochastic integrations in physical sciences, it has been found, however, that the scalar function V is in general not the potential function appearing in the steady state distribution[16], that is, the steady state distribution cannot in general be written as $\exp(-V(\mathbf{q})/\epsilon)$. Only when the noise approaches to zero ($\epsilon \to 0$), V becomes a reasonable approximation to the potential function such as in Eq.(5).

To our knowledge there is no prior discussion on the associated the magnetic field and dissipation similar to what done in the form of Eq.(3) and (5), as well as in section II.

VI. CONCLUSIONS

Based on the demonstrations in section II-IV we conclude that the potential function can coexist with limit cycle. Such a potential function determines the final steady state distribution according to a Boltzmann-Gibbs distribution and the potential gradient drives the dynamics. This may provide an important step in the explicit justification of the existence of potential landscape in complex dynamics. In regarding to computing by attractors in neural networks or the adaptive landscape in evolutionary biology, the present article shows that the Hopfield potential exists for periodic attractors and the Wright adaptive landscape is both a metaphor and quantitative concept.

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