Interactive Middle School Courseware on Abstract Reasoning Skills

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Abstract: Quantitative reasoning skills are a fundamental tool in many fields, ranging from Mathematics to Engineering and from Business to Rhetoric. However, quantitative reasoning is almost never taught as a course, except in the context of other disciplines, such as Mathematics or Physics. We introduce basic elements of a course in reasoning, such as the definition of a concept and the definition of a strategy and study the response of the students. We apply the definitions to algebraic proofing. We conceptualize algebraic concepts, i.e. we name each concept, define its range of applicability and illustrate each concept with typical examples. We also conceptualize strategies for proofing. We find that a diverse population of female middle school students readily accepts this approach and achieves proofing skills on a level which is comparable to university freshmen.

1. Introduction

Multiple step algebra proofs are one of the greatest challenges for middle schools students. The written records of a set of middle school students suggest that they have difficulties formulating their reasoning in writing. Thus, if the number of required steps exceeds what they can easily memorize, their performance is marginal. Students with reasonable writing skills can show their work (see Fig. 1). But even then they struggle to reproduce a strategy and a clear layout of their solution. This problem is similar to writing essays in English, where poor hand writing and organizational skills can be a severe burden for students. For these cases, the problems can be overcome with word processors such as Microsoft Word.

There is a large set of interactive software to teach Math and Science. However, this courseware is typically not designed to teach and test free response multiple step reasoning skills (Hubler & Assad 1996, Raineri et al. 1997). A first step to overcome this problem is the conceptualization of the courseware (Durak & Hubler 2001). Conceptualization means that the courseware provides for each axiom and for each algorithm a name, a definition in terms of less abstract concepts, a range of applicability, and some typical applications.

In this paper we study the conceptualization of strategies, in particular strategies for proofing algebraic statements. We use www-based software which allows students to describe their proofing strategy in English. This reasoning software parses the meaning of their statements and executes them. We find that a group of diverse middle school students readily accepts the software and that the software empowers the students to carry out traditional two column proofs far beyond the middle school level. In addition, the students start to differentiate between different reasoning strategies. We observed that the students use their improved reasoning skills in social interactions and in other subject areas. We explore the conceptualization of reasoning strategies in other fields.

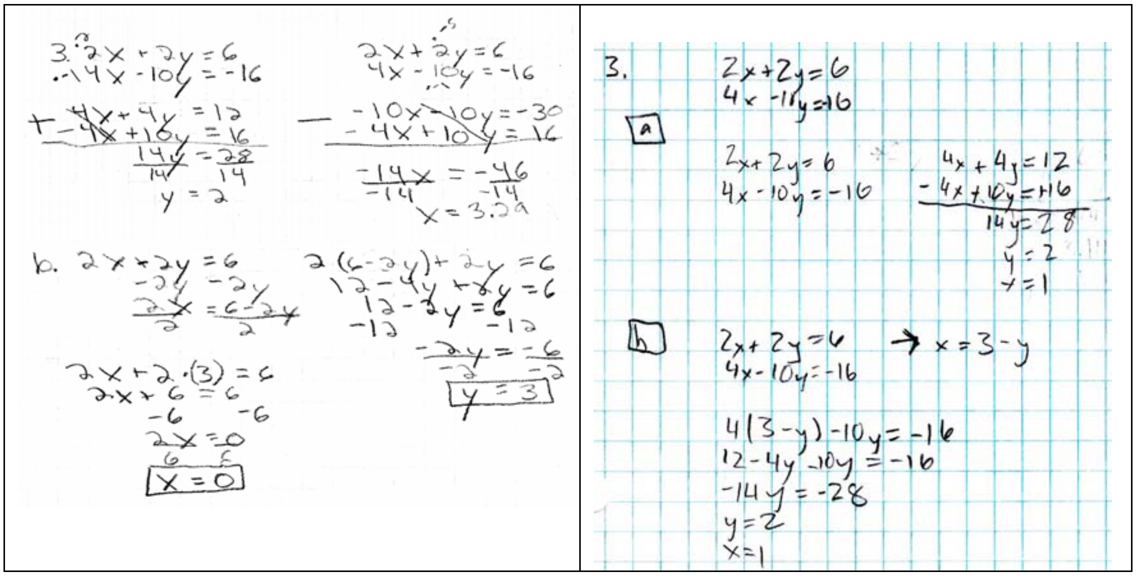


Figure 1: Middle School Level Student Work of a B-Level Student (left) and an A-level Student (right). The given problem is: "3. Given: 2x+2y=6, 4x-10y=-16. Solve for x and y with (a) the linear combination method and (b) the substitution method." The strategy and layout of the A-level student matches the methodology taught in the course. The other student deviates from the strategy and uses a different layout.

2. Definition of a Concept

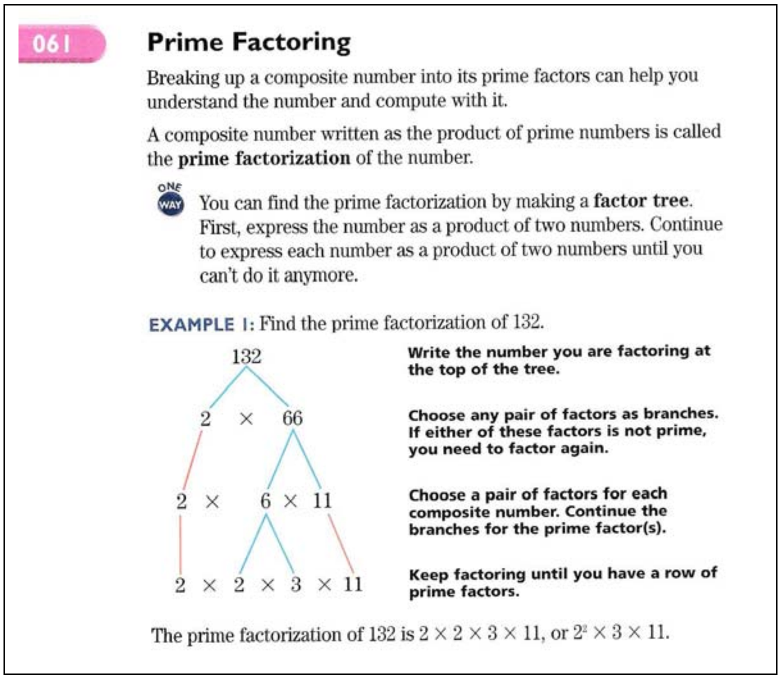
A concept has the following seven ingredients:

1. A **descriptive name**, such as "Multiplication Property of Zero", or "Linear Combination Method";
2. An **identification number**, such as "Law 7" and a **logo**;
3. A **description of the context** in which the concept is typically used, such "Write-the-given-expression is used to start the proof.";
4. A **definition** of the concept in terms of sub-concepts, prerequisite concepts, and common knowledge. This can be a relation between these concepts or a list, such as "Multiplicative Identity Axiom: x*1=x" or "Physics: Mechanics, Thermodynamics, Astronomy …" ;
5. A **range of applicability** identifies when the concept is true and useful. Further synonyms and opposites are introduced. Examples: "The weight force is W=mg for object on the surface of the earth." , or "For any real number x …";
6. Several **single concept examples** use the concept all by itself, such as "What is the weight force on a 2kg object?"
7. Several **multiple concept examples** use the concept plus one or several substitutions. They can also include translations to other fields and common knowledge examples.

Fig. 2 shows a good example of a definition of a concept. The definition is rather brief, but includes the key ingredients of the above list.

3. Conceptualization of the Laws of Algebra

Middle school students tend to complain that they do not know which laws they are allowed to use, when they have to do a proof. We number the laws of algebra and indicate clearly which laws can be used for the proof. This also reduces the set of possible solutions and reduces the likelihood of circular proofs. In addition to the identification number, each law has a descriptive name. The student can refer to the concept both with the descriptive name and with the identification number. Fig. 3 shows the definition of a concept. This brief hand-book-style definition is complemented with a detailed definition, the textbook-style definition, which includes all the ingredients of a concept.



061

Prime Factoring

Breaking up a composite number into its prime factors can help you understand the number and compute with it.

A composite number written as the product of prime numbers is called the **prime factorization** of the number.

ONE WAY You can find the prime factorization by making a **factor tree**. First, express the number as a product of two numbers. Continue to express each number as a product of two numbers until you can't do it anymore.

EXAMPLE 1: Find the prime factorization of 132.

The prime factorization of 132 is $2 \times 2 \times 3 \times 11$, or $2^2 \times 3 \times 11$.

Figure 2: The Definition of the Concept “Prime Factoring” in the Mathematics handbook “Math on Call” (Kaplan 1998). “Prime Factoring” is the name. “061” is the identification number and the graph of the “factor tree” serves as a logo. The first sentence describes the use context. The “One Way” symbol indicates the definition of the concept in terms of sub-concepts. Further a single concept example is given. There are no multiple concept examples. The authors omitted multiple concept examples since this book is considered a companion to a regular text book. Still, multiple concept examples would improve the definition of the concept “Prime Factoring”.

Laws: Law 1 is "a+0=a". Law 2 is "a*0=0". Law 3 is "a*b=b*a". Law 4 is "a+b=b+a". Law 5 is "(a+b)*c=a*c+b*c". Law 6 is "(a+b)+c=a+(b+c)". Law 7 is "(a*b)*c=a*(b*c)". Law 8 is "a*1=a". More details	Law 6 is the **associative axiom for addition**. It means "(a+b)+c=a+(b+c)". This law applies for any real number a, any real number b, and any real number c. Example: Write the given equation: x=(a+2)+3 Use law 10 to replace (a+2)+3: x=a+(2+3)

Figure 3. A Conceptualization of the Laws of Algebra. The identifier and the logo of a set of laws of algebra (left). Here the formula is used as a logo. On the right is a brief definition of Law 6. The descriptive name is “associate axiom of addition”. The second sentence provides the range of applicability. The example shows a typical use context.

4. Conceptualization of the Commands

Typical two column proofs are a sequence of equations or expressions accompanied by a sequence of commands, such as “Commute x and y.” or “Do the arithmetic.” The students are often uncertain on the proper wording of these commands. We introduce a set of commands and a set of synonyms (see Fig. 4), such as “Flatten parenthesis.”, and “Remove parenthesis.”, or “Use law 3 to replace a*b.”, and “Commute a*b.” This limited set of commands helps the students to identify a proper instruction.

Commands: 1: Write the given equation. 2: Memorize this law. 3: Remove parenthesis. 4: Set parenthesis. 5: Do the arithmetic. 6: Add *number*. 7: Multiply by *number*. 8: Use the symmetric axiom of equality. 9: Use law *index*. 10: Replace *name*. 11: Replace *name* by *other-name*. 12: Replace *name* by *other-name*. 13: Replace *name* with law *law*. 14: Replace *expression* with law *name*. More details	8: **Use the symmetric axiom of equality.** Exchange the two members of an equation. Example: Use law 3. a*(b+c)=a*b+a*c Use the symmetric axiom of equality. a*b+a*c=a*(b+c)

Figure 4. A Conceptualization of the Commands. The identifier and the descriptive name of a set of instructions for proofs (left). On the right is a brief definition of command 8. The descriptive name is “Use the symmetric axiom of equality”.

5. Conceptualization of Strategies

There are infinitely many ways to proof a true statement. However if the set of laws is limited, and the number of steps is limited, then the number of different proofs is finite. This makes it possible to define the shortest proof and to distinguish between different proofs. We call a proof with a minimum number of steps an irreducible proof. A given problem may have two irreducible proofs which use the commands in different order. We call these proofs equivalent. However if two irreducible proofs use different commands then we call them two different proofs and name them. Solving equations algebraically is one of the most common ways of proofing. The linear combination method and the substitution method are two different strategies of solving a set of linear equations with two unknowns.

We introduce the following strategies for proofing: (i) start a law with a similar pattern of operations and try to reach the target statement with substitutions, (ii) expand one member of the target equation and regroup the expression until it matches the other member of the target equation and (iii) proof by induction. We used the following strategies for solving linear equations: (i) isolate the unknown, (ii) substitution method, and (iii) linear combination method.

6. Software Implementation

We use the CyberProf™ parser and computer algebra tool to implement the commands. The students enter the command in an HTML input box and CyberProf™ parses and executes the command and inserts the result in a new HTML page. The commands imitate typical text book instructions or teacher’s instructions. These commands describe the proof on a conceptual level. Fig. 5 shows a typical problem. The students start out with grade level appropriate problems and then are guided to increasingly more advanced proofs.

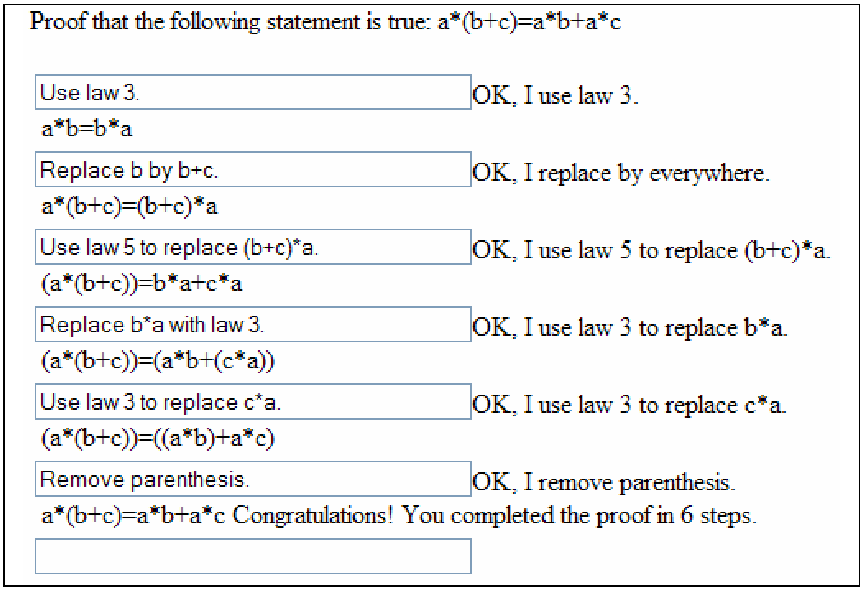


Figure. 5: WWW Online Implementation of Computer Assisted Reasoning Software. The students enter commands in the input boxes. The computer parses the command and responds next to the input box. Then the computer executes the command and writes the result below the input box.

Since the proofing software is implemented on the internet the students can access the software during class and at home. The students can transform equations, proof new laws, add them to the list of laws, and use these laws to simplify expressions. Further, they can use this computer assisted reasoning system to solve linear equations. In contrast to other computer algebra systems such as Mathematica or Mathlab, this system understands instructions in English. Therefore, the system can be used by a much larger fraction of the population and by younger students.

7. Student Response

There is only colloquial evidence of the impact of the computer assisted reasoning system. The system was tested in a small diverse class of female middle school students. All students finished grade level appropriate proofs with up to ten steps. The top students need about 1min/step whereas the slowest students need about 2min/step. The biggest impact was on the underperforming students. On paper proofs they have a hard time to get started and they tend to get lost if the proof requires more than three steps. In contrast, immediate feedback from the computer assisted reasoning system helps them to stay on track. The well performing students like to compete in finding the shortest proof and they are delighted when they create a unique proof. The students occasionally complain because the system forces them to justify each step, whereas, on a paper proof, they may get away with skipping a few steps. Furthermore, a professional outside evaluator of the G-K12 NSF grant noticed that the students started to use their reasoning skills in social interactions outside the classroom. The system enables top students to perform college level proofs. One student summarized it like this: “Without this system I could never have done that proof”. The reasoning software has a strong impact on paper proofs. Fig. 6 shows typical student work, after the students have used the reasoning software for five times 15 minutes. In contrast to the work shown in Fig.1, the students are comfortable justifying each step. We have encouraged the students to conceptualize non-algebra problems, such as mathematical puzzles and proofs in geometry. A strategy to conceptualize a new set of task includes the following steps:

1. What are the objects? In middle school algebra, the objects are expressions and equations.
2. What are permissible operations and commands? In algebra, permissible commands are string replacements which keep the value of an expression the same or which keep an equation true. These

commands are derived from equivalency relations, the laws of algebra. An example for a command is: "Use law 3 to replace a*c by c*a."

3. What are promising strategies, i.e. which sequence of commands have been successfully applied to similar tasks? For instance "isolating the unknown variable" is a promising strategy for solving linear equations.

The students readily accept this approach. It helps the students to record their approach in writing and to solve more advanced problems.

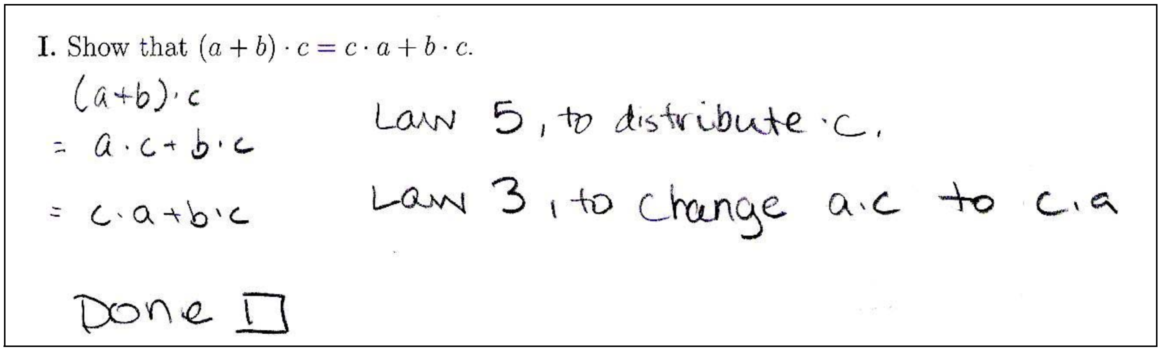


Figure 6: Student Work after Five 15 minute Sessions with the Online Reasoning System. The students are getting used to justifying each step. The layout of the student's work becomes more two-column style. This style is the preferred style in the students' textbook. The layout of the reasoning software is an alternating sequence of commands and equations or expressions (see Fig. 5).

8. Discussion

We have introduced a computer assisted reasoning system in the middle school classroom. The system requires a conceptualization of the methods and strategies, but helps the students to perform advanced reasoning tasks. This approach can be used in other areas such as Chemistry, Physics, or Economics. Usually textbooks are partially conceptualized and the step from a partial conceptualization to a full conceptualization is small. However fully conceptualized strategies can be computerized and included in computer assisted reasoning systems. Standard reasoning systems and computer algebra systems understand only a special code, whereas this reasoning system understands English and is available on the World Wide Web. Thus it can be used not only by middle school students but by a large fraction of the population.

Acknowledgement

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