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Reconstructing equations of motion from experimental data with unobserved variables

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We have developed a method for reconstructing equations of motion for systems where all the necessary variables have not been observed. This technique can be applied to systems with one or several such hidden variables, and can be used to reconstruct maps or differential equations. The effects of experimental noise are discussed through specific examples. The control of nonlinear systems containing hidden variables is also discussed.

I. INTRODUCTION

Prior to the 1980s, researchers had always assumed that to study the dynamics of nonlinear systems with many degrees of freedom, time-series measurements of all the variables, or derivatives thereof, were necessary to generate state-space representations of the dynamics. For experimental systems, derivatives are particularly difficult to employ due to the noise problems. In 1980, Packard *et al.*[1] and Ruelle noted that a state-space representation of the dynamics could be reconstructed from a single time series through the use of delay coordinates. This delay-coordinate reconstruction would then be topologically equivalent to the dynamics of the true system. Whitney had shown much earlier[2] that any compact manifold with dimension m can be embedded in R^{2m+1}. Takens extended this[3] in proving that an embedding can be obtained for any system from only a single time series by using $2m+1$ delay coordinates.[4] While this combination of ideas thus far has been extremely useful in studying nonlinear systems, several difficulties arise in their application that we hope to address with an alternative method for reconstructing these *hidden* variables.

The most obvious difficulty in using delay coordinates is the issue of interpreting the results physically. If one is concerned only with forecasting, the method used for the modeling is irrelevant, as long as it works. Successful modeling techniques based upon delay coordinates and/or partitioning the state space to generate local fits have been developed.[5–7] However, relating these models back to physical principles or existing theories is often difficult if not impossible. Another important consideration is that Takens's theorem does not apply to systems with noise, i.e., experimental data. In the case of (noisy) experimental data, one must define what precisely is meant by an embedding. We will call an embedding any representation for which any two observations $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ within σ, the noise amplitude, of each other are followed by $\mathbf{X}_1(t+\delta t)$ and $\mathbf{X}_2(t+\delta t)$ within the propogated error of each other. Another way of stating this could be that for any region of the state space, the variance of the succeeding values is minimized. Casdagli has reported progress in extending Takens's theorem to find optimal embeddings in the low noise limit,[8] but at present there exists no general guarantee that a given reconstruction will be an embedding in the presence of noise.

In order to create a modeling technique in which existing information can be incorporated, the resulting model can be interpreted physically, and which is reasonably stable to noise; we base our technique upon the *flow method* developed by Cremers and Hübler[9] and Eisenhammer *et al.*[10] that is similar to that of Crutchfield and McNamara.[11] The flow method is a procedure for reconstructing a set of coupled maps (CM's) or ordinary differential equations (ODE's) from a trajectory of the system in state space. We will show that this may be easily adapted to the presence of hidden variables.

In Sec. II, we will review the flow method and demonstrate how hidden variables can be incorporated into this framework. Some limitations of these techniques are also addressed. Section III provides details on the implementation of our hidden variables technique with specific examples of its application to simulated data. The treatment of noisy experimental data is also discussed. The use of this modeling procedure in conjunction with nonlinear control theory to control systems where some variables are hidden from observation, control, or both is discussed in Sec. IV.