

Transformation to Normal Time for the Calculation of Poincaré Maps in a Closed Form

A. Wandelt, W. Eberl*, A. Hübler, E. Lüscher
Physik-Department E13, D-8046 Garching

Abstract

The calculation of Poincaré maps can be simplified by introducing a special rescaling of the flow vector field, namely the transformation to normal time. If that is done in an appropriate way, one can calculate the Poincaré map analytically.

1 Introduction

Oscillators with marked nonlinearity represent good models in various fields of physics. In many cases these systems have a smooth short-time behaviour but complex long-time dynamics. Mostly these equations cannot be solved in a closed form. On the other hand these systems can be effectively described by stroboscopic maps or Poincaré maps. For several problems it was shown that the dynamics of the continuous system is strongly correlated to the dynamics given by the corresponding map [2].

In this paper we present a method to calculate the parameters of a Poincaré map from the corresponding differential equation analytically. Recent studies by Wackerbauer et al. [1] have shown that it is possible to calculate a stroboscopic map from nonlinear ordinary differential equations in a closed form. However, in many cases the stroboscopic maps are too complicated for an effective description of the dynamic system. This is due to the coupling of frequency and amplitude of the oscillating system. This coupling is a general feature of nonlinear oscillators [3]. Due to this, Poincaré maps are independent of the amplitude frequency coupling and therefore for systems with a large amplitude frequency coupling sometimes essentially smoother and simpler than stroboscopic maps.

Recently it has been shown that for a large variety of oscillators an appropriate transformation, the normal time transformation, can be used in order to decouple amplitude and frequency. In this case stroboscopic maps are special Poincaré maps and are much less complicated. It should be now possible to calculate Poincaré maps by using algorithms designed for calculating stroboscopic maps and an expansion of stroboscopic maps with respect to the initial condition of the state variables should converge quickly. However this is not the case since the scaling function proposed in [4] was not continuous.

In this paper we present a new transformation for the decoupling of amplitude and frequency where the transformed flow vector field remains continuous. For linear oscillators amplitude and frequency are decoupled. Hence we propose a transformation that forces the angular velocity in phase space to be equal to that of a damped harmonic oscillator.

2 Transformation to normal time

For a two dimensional system of ordinary differential equations $\dot{\vec{x}} = \vec{F}(\vec{x})$ the angular velocity in phase space is given by

$$\dot{\varphi} = \frac{F_2 x_1 - F_1 x_2}{x_1^2 + x_2^2} .$$

For higher dimensional systems, x_1 and x_2 are two particular state variables that must be chosen conveniently. If it is possible to determine an appropriate Poincaré surface, it should also be possible to select the corresponding variables.

Multiplying the flow vector field by a scalar function $s(\vec{x})$ does not change the geometry of the trajectories. Choosing $s = 1/|\dot{\varphi}|$ one yields a constant angular velocity for the rescaled system. The disadvantage of that choice is that there is a singularity at $\vec{x} = \vec{0}$. In order to calculate a Poincaré map it is only necessary to set the period of the system to a constant value.

Therefore we suggest to choose $s = |\dot{\varphi}_H/\dot{\varphi}|$, where

$$N_H = \frac{x_2^2 + \gamma x_1 x_2 + \omega^2 x_1^2}{x_1^2 + x_2^2}$$

is the angular velocity of a damped harmonic oscillator with damping constant γ and frequency $\omega \cdot \sqrt{1 - (\frac{\gamma}{2})^2}$. To obtain a continuous scaling function one must choose γ and ω as follows:

$$\omega^2 = -\frac{1}{m} \left(\frac{\partial F_2}{\partial x_1} \right)_{\vec{x}=\vec{0}} \quad \gamma = \frac{1}{m} \left(\frac{\partial F_1}{\partial x_1} - \frac{\partial F_2}{\partial x_2} \right)_{\vec{x}=\vec{0}}$$

$$\text{with } m = \left(\frac{\partial F_1}{\partial x_2} - \frac{1}{2} \frac{\partial^2 F_2}{\partial x_2^2} \right)_{\vec{x}=\vec{0}}$$

Fig. 1 shows the effect of the transformation applied to the Rössler-System

$$\dot{x}_1 = -x_2 - x_3; \quad \dot{x}_2 = x_1 + ax_2; \quad \dot{x}_3 = b + x_3(x_1 - c).$$

3 Solving the rescaled equation

Using a numerical method like in [4] one can obtain the Poincaré map well approximated by a power series with only few nonzero coefficients:

$$x_{1_{n+1}} = \sum_{i,j=0}^{i+j=4} a_{i,j} x_{1_n}^i c^j$$

where x_{1_n} = n th cutting point of the trajectory with the Poincaré surface and $a_{i,j}$ represents a calculated coefficient.

The dependence of the map on x_3 is neglectible because the calculated coefficients are much smaller than the leading ones.

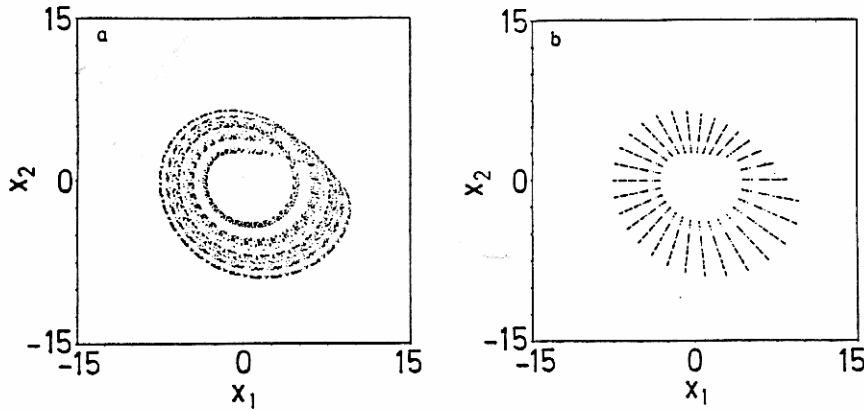


Figure 1: Discrete trajectories of the Rössler attractor projected in the (x_1, x_2) -plane. The left picture was calculated from the original equation with $a = 0.2$, $b = 0.2$, $c = 4.75$. The time between two points is $\Delta t = \frac{2\pi}{30}$. The right picture was calculated with the rescaled equation for the same parameter values by using the exact rescaling $s = \frac{x_1^2 + ax_1x_2 + x_2^2}{(-x_2 - x_3)x_2 - (x_1 + ax_2)x_1}$ and $\Delta t = \frac{2\pi}{30}(1 - 0.25a^2)^{-\frac{1}{2}}$.

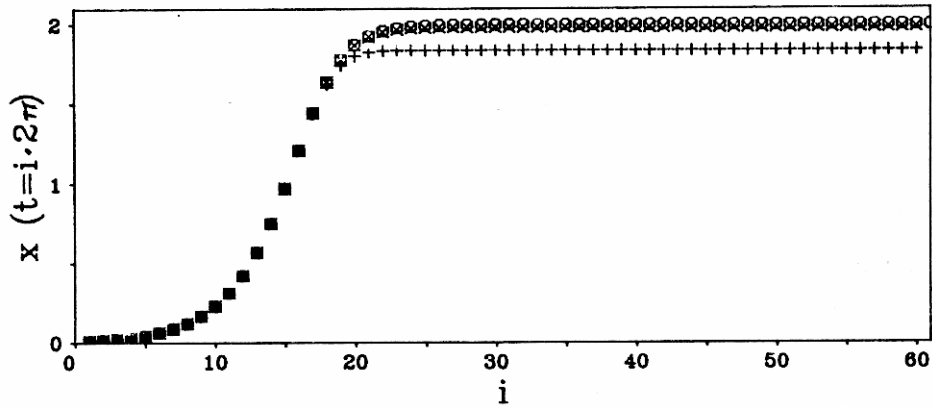


Figure 2: 60 numerical integrations over $\Delta t = 2\pi$ of the rescaled system of the van der Pol Oscillator $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + 0.1(1 - x_1^2)x_2$; (x): using the exact rescaling s ; (+, o): using the approximated $s = -1 + 0.1x_2 + 0.1(x_1 - 1)x_2 - 0.04(x_1 - 1)x_2^2 - 0.04(x_1 - 1)^2x_2^2$ with the expansion point $(1, 0)$; (+): setting $x_2 := 0$ after each integration; (o): $x_1 := \sqrt{x_1^2 + x_2^2}$ and $x_2 := 0$ after each integration.

$i_1 j_\mu$	$a_{i_1 j_\mu}$	$b_{i_1 j_\mu}$	$c_{i_1 j_\mu}$
0 0	1.8213	2.6885	1.016
0 1	-5.5401	-1.8331	-0.089
0 2	3.2582	1.5343	0.490
0 3	-0.5292	-0.0383	
0 4	-0.0521		
1 0	-1.2041	1.6430	1.137
1 1	1.9245	2.1822	1.028
1 2	-0.0138	0.1769	
1 3	-0.3360		
2 0	0.1712	1.1403	0.615
2 1	0.1313	0.2795	
2 2	-0.4691		
3 0	-0.0442	0.1107	
3 1	-0.2769		
4 0	-0.0648		
Δs_m	$3.9 \cdot 10^{-3}$	$3.8 \cdot 10^{-2}$	$5.2 \cdot 10^{-1}$
Δs_l	$9.2 \cdot 10^{-3}$	$7.7 \cdot 10^{-3}$	$1.5 \cdot 10^{-1}$

Table 1: Coefficients of the Poincaré map for the Rössler system with $a = b = 0.2$

Another way is to use an analytical method that has already been developed [1]. For this algorithm it is necessary to represent the flow vector field as a polynomial. This can be reached by expanding s into a Taylor series around an appropriate point on the Poincaré surface. The deviation due to neglecting higher terms than second order are relatively small (Fig. 2).

We'd like to thank B. Buchberger, M. Singer and all the people at RISC Linz helping us to use Computer Algebra, and A. Hayd, M. Maurer and P. Meinke (MAN) for valuable discussions.

* Part of PhD thesis

4 References

- [1] R. Wackerbauer, W. Eberl, A. Hübler, E. Lüscher, H.P.A. **61**, 228 (1988)
- [2] J. Guckenheimer, P. Holmes: *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*, New York: Springer 1983
- [3] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*, New York 1979, p.54
- [4] M. Kuchler, W. Eberl, A. Hübler, E. Lüscher, H.P.A. **61**, 232 (1988)