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Extremal properties of dendritic patterns : biological applications

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Abstract

Recent experiments indicate that stable agglomerates of metallic balls in an electric field can have a fractal geometry and a dendritic structure. Further it has been shown, that these patterns satisfy a variational principle. We compare the geometry of these optimal patterns with dendritic structures of biological systems, e.g. the nervous system, roots. We discuss the relationship between the variational principle and the function of these roots.

1 Introduction

Dendrites are ramified, simply connected patterns. They often are self-similar in a certain range of scales/1/. Particularily for living beings they seem to be of importance. The dendritic geometry is the basic geometry of roots, ramified nerves, arteries, lymphvessels, etc.. There is also a large class of open physical systems, which provide dendritic structures, e.g. breaklines of hard materials, river systems, electric lightnings. Over the past several years, the Hele-Shaw system/2/, dielectric breakdown/3/, solidification of undercooled melts/4/, growth by electrochemical deposition/5/, precipitate growth in supersaturated systems/6/, and the flow through porous media/7/ have been investigated experimentally and theoretically. Recently it has been shown, that a special class of dendritic structures can represent stationary, stable states of an open system, and that these structures satisfy a variational principle. The aim of this paper is to illustrate, that the optimization of the corresponding quantity might be useful for living beings.

2 Dendritic patterns which satisfy a variational principle

One of the simplest physical systems that generates optimal dendritic patterns has been presented by Georgii et al./8/. The experimental set-up essentially consists of a cylindrical cell , made up by a 3 mm layer of castor oil (because of its high dielectric constant) within an acrylic dish of 114 mm diameter and a 47 mm layer of air above . The inner perimeter of the dish is equipped with a grounded metal ring electrode. The potential between a metallic tip, at 47 mm above the center of the oil surface, and this ring is 20 kV. Charges are sprayed quasi homogenously upon the oil surface. The dish contains N metallic bearing balls at first distributed randomly. Under the influence of the electric field they form a dendritic structure to transport the charges to the metal ring /9/. Since the layer of oil is thin, compared to the diameter of the cell, we use a two dimensional model and investigate the electric field between the balls. The potential energy W results from

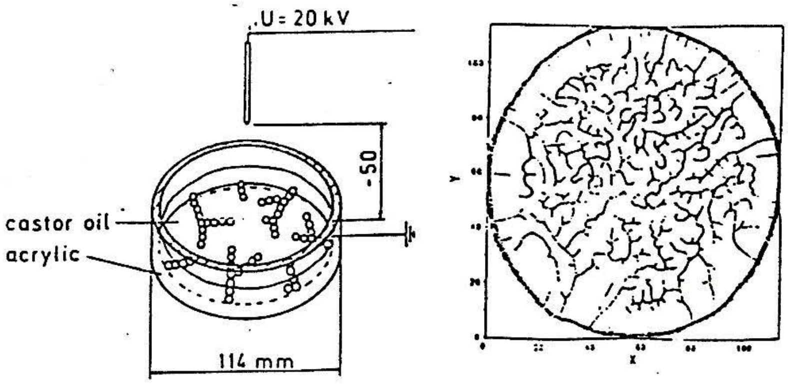

Figure 1: a) Experimental set-up b) Picture of a dendrite

$$W = \int_{Oil} \varepsilon(\nabla\Phi)^2 d\vec{x} \tag{1}$$

where ε is the dielectric constant of the oil and the electric potential Φ is calculated from

$$\sigma\triangle\Phi(\vec{x},t) = S(\vec{x},t) + \frac{\partial\varrho(\vec{x},t)}{\partial t} \tag{2}$$

$\vec{x}$ and t are position and time respectively. The conductivity of the oil is σ, ϱ is the charge density at the surface and S counts the number of charges which are sprayed, per unit time, onto the surface at position $\vec{x}$. To derive equation (2) the continuity equation for the charges has been used under the constrain that Ohm's law is valid. The source of charges S of equation (2) is assumed to be constant. The metallic balls, which are assumed to be ideal conductors, are represented by small circular areas centered around position $\vec{q}_i (i = 1, 2, ..., N)$. We assume further that the dynamics of the balls can be modelled by

$$\gamma\dot{\vec{q}}_i = -\nabla_{\vec{q}_i} W \tag{3}$$

in which γ is the friction constant. If the electrical relaxation time of equation (2) is very small compared to the mechanical relaxation time resulting from equation (3), in addition to the boundary conditions the electric potential $\Phi(\vec{x}, t, \vec{q}_i(t))$ is a function of $\vec{x}$ and of the positions $\vec{q}_i$ of the balls only. If the boundary conditions for the current are kept fixed, the electric potential Φ is slaved /9/ by the positions $\vec{q}_i$ of the balls , i.e. $\Phi = \Phi(\vec{x}, \vec{q}_i(t))$. In this case W is a Lyapunov function of the system. Therefore the local and global minima of W are stable stationary states /9/. Since the dissipation P is proportional to the potential energy W, the system approaches a state with minimal dissipation, i.e. a minimal production of heat. It has recently been shown experimentally that the geometry of the stable patterns satisfies the scaling relations

$$W \sim M^{-D_M} \tag{4}$$

$$W \sim N^{-D_N} \tag{5}$$

where D_M and D_N are dependent on the geometry of the experiment and of the current traversing the system. M is the number of the tips of the structure and N is the number

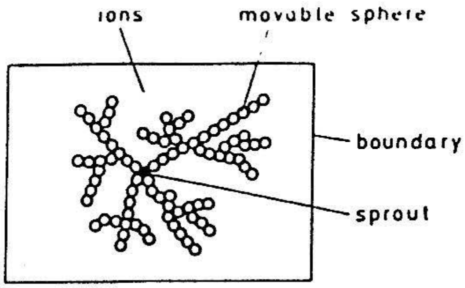

Figure 2: Modell of a root system

of directly or indirectly grounded balls, D_M and D_N are independent of M or N. Further *Merté* et al./9/ have shown that the optimal patterns are ramified and simply connected. Obviously these are some of the properties of plants. In the following chapter we will present a mathematical link between the properties of roots and the optimal dendritic structures of the physical system.

3 Optimal structures for roots

The purpose of the root system of plants is the mechanical support of the plant as well as the absorbtion of ions and water. In the following we focus on the latter function and use a simplified, two dimensional model of the root system (fig.2). The black dot in the center of the container represents the connection between the root system and the plant and is assumed to be an ideal absorber for ions. J quantifies the amount of ions which are homogenously spread into the container per unit time. N spheres are movable and represent the system of roots. Therefore we assume that the diffusion constant D_i for the ions inside the spheres is essentialy higher than the diffusion constant D_s in the surrounding medium. Further we assume that ions can not diffuse across the boundary of the container. Due to these assumptions we have the following relation for the concentration ϱ of the ions

$$\frac{\partial \varrho(\vec{x},t)}{\partial t} = D\, \nabla^2\, \varrho(\vec{x},t) + J \tag{6}$$

W_b is given by

$$W_b = \left| \int_{cell} (\nabla \varrho)^2 d\vec{x} \right| = \frac{J}{D_s} \bar{\varrho} \tag{7}$$

where $\bar{\varrho}$ is given by

$$\bar{\varrho} = \int_{cell} \varrho d\vec{x} \tag{8}$$

W_b depends on the positions of the balls and on J. If the pattern of spheres represents the stable state of the physical system of chapter 2, and J is constant, then W_b is minimal. A small value of W_b might be an advantage for the plant, if it competes with other plants, or other absorbers are in the surrounding medium. If we assume that a small value of W_b is an advantage for a plant, then we can conclude from chapter 2 that optimal root systems satisfy the following relation

$$W_b \sim M^{-D_M} \sim N^{-D_N} \tag{9}$$

M is the number of tips and N is the number of balls with direct or indirect contact to the black dot (the total mass of the root system). D_M and D_N are factors depending on the geometry of the experiment, as well as on the ion current traversing the system, and are independent of M.

4 Discussion

Of course the root system of real plants has to satisfy other properties too, e.g. resistivity against diseases. Since the root system of a huge variety of plants seems to be closely related to optimal physical dendrites, we assume that the optimization of W_b is an essential advantage for most plants. Since the basic equations which have been used in the physical system and the biological model are first order approximations for the general transport processes, the argument presented in this paper might be the basis for the explanation of the ramified systems of nerves (collection of information from the surroundings), arteries (distribution of oxygen etc.), and other dendritic structures in biological systems.

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