ANALYTICAL REPRESENTATION OF STROBOSCOPIC MAPS OF ORDINARY NONLINEAR DIFFERENTIAL EQUATIONS

R. Wackerbauer, W. Eberl, A. Hübler*, E. Lüscher,
Physik-Department, Technische Universität München, D-8046 Garching

Abstract: Dynamical systems, represented by a system of ordinary differential equations, are described by analytically calculated stroboscopic maps.

1. Introduction

Based on the slaving principle /1/ a large class of complex systems in various fields of physics can be described by a low dimensional system of ordinary differential equations (deq.). The dynamics of nonlinear systems with complex long-time-behaviour and smooth short-time-dynamic can be described more easily by means of a stroboscopic map than by the original differential equation. It will be shown in this paper that the corresponding stroboscopic maps can be obtained analytically by means of a series expansion.

2. Calculation of the Stroboscopic Map

Proceeding from a n dimensional system of ordinary deq. $\dot{\underline{x}}=\underline{f}(\underline{x}(t))$, where $\underline{x}(t)$ is the n dimensional state vector, the stroboscopic map $\underline{S}$: $\underline{S}(\underline{x}(t),T)=\underline{x}(t+T)$ is calculated. Only those systems are considered, whose flow vector field $\underline{f}$ can be represented as a polynom of degree g. $\underline{S}$ maps the state vector $\underline{x}(t)$ at time t into the state vector at time t+T. The dynamic of the stroboscopic map is obtained by $\dot{\underline{S}}(\underline{x}_0,T)=\underline{f}(\underline{S}(\underline{x}_0,T))$, whereas $\underline{S}(\underline{x}_0,T=0)=\underline{x}_0:=\underline{x}(t=0)$ is the initial condition of the system. As representation of the solution of the stroboscopic map a series expansion is chosen in powers of the initial condition:

$$S_i(\underline{x}_0,T) = \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} a_{i,\underline{k}}(T) \cdot \prod_{j=1}^{n} x_{0j}^{k_j} \qquad (1)$$

where $\underline{k}$ is a n-dimensional index vector, x_{0j} is the j-th component of $\underline{x}_0$.

The dynamic of the system is expressed in terms of time dependend mode amplitudes $a_{i,\underline{k}}$. Comparing the coefficients of the series expansion in the deq. for the stroboscopic map one gets a system of ordinary deq. of infinite dimension for the mode amplitudes.

Due to the special initial condition only n modes $a_{i,\underline{k}}$ are not zero for T=0. The radius of convergence of the stroboscopic map $\underline{S}$ is infinite for T=0. There are nonlinear systems with known analytical solutions, i.e. for these systems the series expansion (1) is convergent. If such a solution is not yet known, the mode amplitudes can be estimated either using Gronwall's lemma /2/ or numerically.

The following properties facilitate the solvation of the differential equations for the mode amplitudes. (i) The flow vector field $\underline{f}$ is represented as a polynom, thus the equations for the mode amplitudes are hierarchic, e.g., the mode $a_{i,\underline{K}}$ only depends on modes $a_{j,\underline{k}}$ whereby the components of $\underline{k}$ are less than or equal to the corresponding components of the index vector $\underline{K}$. A proof of this and the following statement will be published elsewhere. The examples of chapter 3 are showing these properties. (ii) If $\underline{x}=\underline{0}$ is a fixpoint of the flow vector field, the deq. of the mode for $a_{i,\underline{K}}$ are reduced to a system of linear deq. with a nonlinear inhomogenity consisting of known modes $a_{j,\underline{q}}$, and $\|\underline{q}\| < \|\underline{K}\|$. Whereby the 1-norm of a n dimensional vector $\underline{v}$ is defined as follows: $\|\underline{v}\| := \sum_{i=1}^{n} |v_i|$

3. Examples

3.1 The van der Pol-oscillator as a two dimensional system

Proceeding from the nonlinear deq. of the van der Pol-oscillator $\ddot{x}-e(1-x^2)\dot{x}+x=0$, whereby e controls the nonlinearity of the system, one gets the two dimensional system of deq., $\dot{x}_1=x_2$, $\dot{x}_2=e(1-x_1{}^2)x_2-x_1$. From this the hierarchic deq. for the mode amplitudes is achieved:

$$\dot{a}_{1,nm} = a_{2,nm}$$

$$\dot{a}_{2,nm} = ea_{2,nm} - a_{1,nm} + e \sum_{i,j,k,l,p,q}^{\infty} a_{1,ij} a_{1,kl} a_{2,pq}\, \delta_{n,i+k+p}\, \delta_{m,j+l+q}$$

The initial condition, $a_{1,10}(T=0)=a_{2,01}(T=0)=1$ and all other modes $a_{1,nm}(T=0)=a_{2,nm}(T=0)=0$, leads to $a_{1,\underline{0}}(T)=a_{2,\underline{0}}(T)=0$ as the solution of the mode amplitudes for $\underline{k}=\underline{0}$; therefore the nonlinear deq. can be reduced to a linear deq. with known nonlinear inhomogenity and this means the stroboscopic map can be solved analytically.

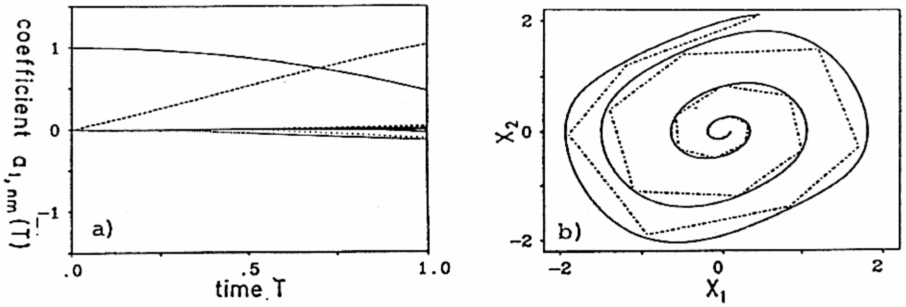

fig.1:a) Representation of the mode amplitudes $a_{1,nm}$ of the van der Pol-oscillator as a function of T (e=0.4, T=1, n,m=0,1,..,5).

b) Trajectories of the van der Pol-Oscillator in the phase space. Full line: numerical integration of the original deq.; dotted line: iteration of $\underline{S}$ using the modes of a).

3.2 The Helmholtz-oscillator as a system with chaotic dynamic

Analogously to 3.1 one can deduce from the deq. of the Helmholtz-oscillator $\ddot{x}+ex+gx+x^2=f\cos(wt)$, with damping constant e, and g depending on the form of the potential, a four dimensional system of first order deq. with a fixpoint at $\underline{x}=\underline{0}$. $\dot{x}_1=x_2$; $\dot{x}_2=-ex_2-gx_1-x_1^2+x_3$; $\dot{x}_3=x_4$; $\dot{x}_4=-w^2x_3$. Then one obtains the deq. for the modes:

$$\begin{pmatrix} \dot{a}_{1,\underline{k}} \\ \dot{a}_{2,\underline{k}} \\ \dot{a}_{3,\underline{k}} \\ \dot{a}_{4,\underline{k}} \end{pmatrix} = \begin{pmatrix} a_{2,\underline{k}} \\ -ea_{2,\underline{k}} - ga_{1,\underline{k}} + a_{3,\underline{k}} \sum_{\underline{l},\underline{m}} a_{1,\underline{l}} a_{2,\underline{m}} \cdot \delta_{\underline{k},\underline{l}+\underline{m}} \\ a_{4,\underline{k}} \\ -a_{3,\underline{k}} \end{pmatrix}$$

Solving this system of linear deq. one obtains a stroboscopic map. The trajectories calculated by analytical integration of the Helmholtz-oscillator are in good agreement with the numerically calculated trajectories. Also the bifurcation diagram can be reproduced with the analytically calculated stroboscopic map. For the calculations in fig.2b only these modes of the stroboscopic map are used, of which the inhomogenity is not vanishing for T=0. The bifurcation diagrams are plotted for some different values of the damping constant e.

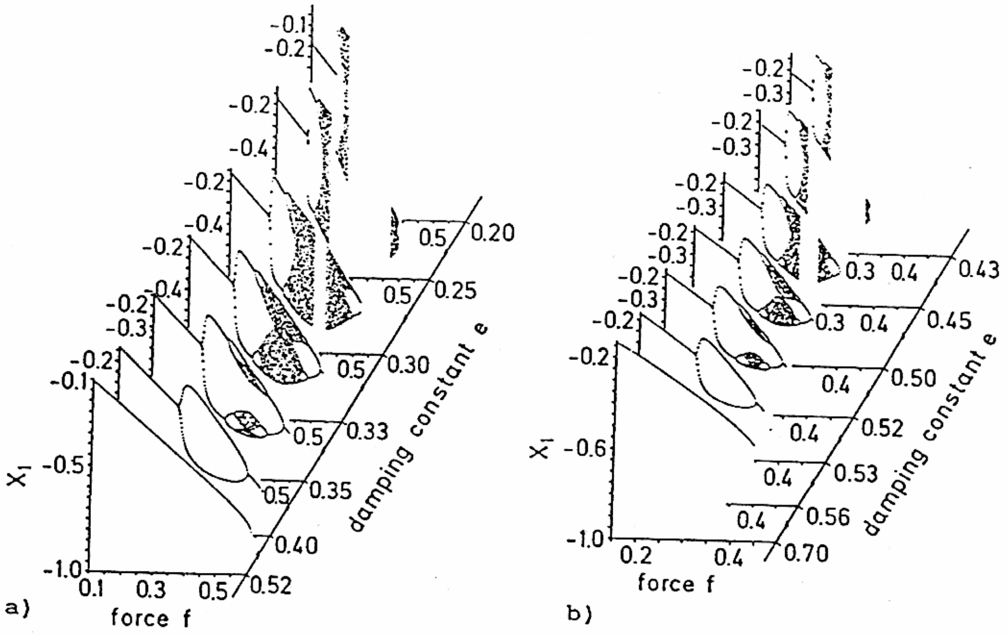

fig.2: Bifurcation diagrams of $x_1(t)$ for $t=2n\pi$ of the Helmholtz-oscillator (g=0.5, w=1) as a function of the force parameter f, calculated

a) using an analytically calculated stroboscopic map $\underline{S}(\underline{x}_0,\pi/2)$

b) by numerical integration of the original deq.

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References

* = part of Ph.D. thesis

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/2/ Guckenheimer,J., Holmes,P.: Nonlinear Oscillations,Dynamical Systems, and Bifurcations of Vector Fields, New York: Springer 1983 p.169