Algorithm for the Determination of the Resonances of Anharmonic Damped Oscillators

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Z. Naturforsch. 42 a, 803–807 (1987); received September 15, 1986 and May 29, 1987

Non-linear damped oscillators can be resonantly driven by an aperiodic force. An algorithm is described which shows how such a driving mechanism can be calculated. This new method paves the way for resonance spectroscopy in non-linear systems.

1. Introduction

In addition to the passive observation of a non-linear oscillator and the description of the measured data with Lyapounow exponents [1, 2], fractal dimensions [3–5], entropies [6], etc. it is possible to charcterize a non-linear oscillator by an active method, namely by determining its response to specific perturbations. The resonances of harmonic systems which are brought about by a sinusoidal perturbation and a systematic variation of the frequency are of major significance. Resonance spectroscopy has proved very successfull in many fields of physics. Huberman and Crutchfield [7] and many other groups (see, for example, [8–10]) have shown that damped oscillators with marked nonlinearity respond to a purely periodic perturbation with complex, in many cases chaotic dynamics. This chaotic response is difficult to characterize, comparatively small, nonresonant. because the driving force and velocity of the oscillator are out of phase [11]. The following approach shall attempt to derive resonant driving mechanisms for damped non-linear oscillators.

2. The Resonant Stimulation of a Nonlinear Oscillator

The non-trivial relationship between eigenfrequency and amplitude is a characteristic feature of a non-linear oscillator. If, for example, a slightly damped oscillator in the potential $V(y) = y^6$ ($y =$ amplitude) is regarded, which is in the neighbourhood of the minimum of the potential, its oscillation frequency is small, but increasing strongly with a larger amplitude of oscillation. An ideal (resonant) perturbation which is in phase with the velocity of the oscillator [11], should start with a small frequency and show in the course of time a more and more increasing frequency. If a model which is described by the same differential equation is numerically integrated, starting from a highly stimulated state, one obtains at first oscillations of high frequency, which slow down in the course of time. Is this set of data reflected in time, one obtains for the frequency exactly that time dependence which is needed for the ideal perturbation. In the following it will be shown analytically that with the help of this trick a resonant stimulation is possible with nonlinear damped potential oscillators, and then ways of application will be shown by giving some numerical examples.

Let us regard nonlinear oscillators of the type

$$-\frac{d^2y}{dt^2} - \eta_1 \frac{dy}{dt} - \frac{dV_1(y)}{dy} = 0 \tag{1}$$

(t = time, η_1 = friction coefficient)

and let us assume that the amplitude y at time $t = 0$ is in the vicinity of a minimum of the potential $V_1(y)$. We call this system, which is to be stimulated, experimental system in contrast to that system (model = mathematical model) which we need for the calculation of the driving force. The driving force is calculated by integrating the differential

* Part of Ph.D. thesis.

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equation of the model

$$-\frac{d^2x}{dt^{*2}} - \eta_2 \frac{dx}{dt^*} - \frac{dV_2(x)}{dx} = 0 \qquad (2)$$

(t^* = time, η_2 = friction coefficient,
x = amplitude, $V_2(x)$ = Potential)

starting from a highly stimulated state at time $t^* = 0$, by reflecting in time, differentiating and rescaling the obtained set of data $x(t^*)$,

$$F(t) = -2\,\eta_2 \frac{dx}{dt^*}, \quad t^* = T - t. \qquad (3)$$

T results from the condition $x(t^* = T) \approx y(t = 0)$.

The stimulated experimental system

$$-\frac{d^2y}{dt^2} - \eta_1 \frac{dy}{dt} - \frac{dV_1(y)}{dy} + F(t) = 0 \qquad (4)$$

then possesses the special solution $y_s(t)$:

$$y(t) = y_s(t) = x(t^*(t)), \quad t^* = T - t$$
$$\text{for} \quad V_1(y) = V_2(y), \quad \eta_1 = \eta_2. \qquad (5)$$

The energy of the experimental system

$$E = \left(\frac{dy}{dt}\right)^2 / 2 + V_1(y)$$

increases for $y(t) = y_s(t)$ continuously. The perturbation is resonant for $y(t) = y_s(t)$, as in this case the velocity of the experimental system and the driving force are in phase [11], i.e. $dy/dt \sim F(t)$.

From the experimental point of view it is of great importance whether this special solution is stable. If it is the case, only a rough knowledge of the initial conditions of the experimental system is necessary for a resonant stimulation. For harmonic systems the stability of the special solution y_s can easily be verified. For anharmonic systems the hypothesis has been numerically verified for several examples (Figs. 1 and 2) [12].

3. Resonance Spectroscopy

If the differential equation of the model and of the experiment differ slightly, the experimental system can only be stimulated with less success or not at all. If the unknown experimental system

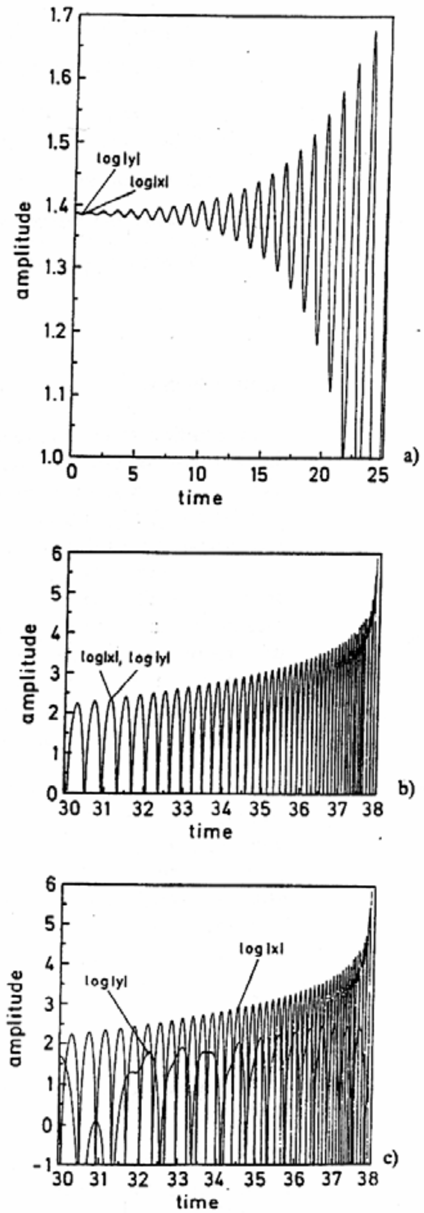
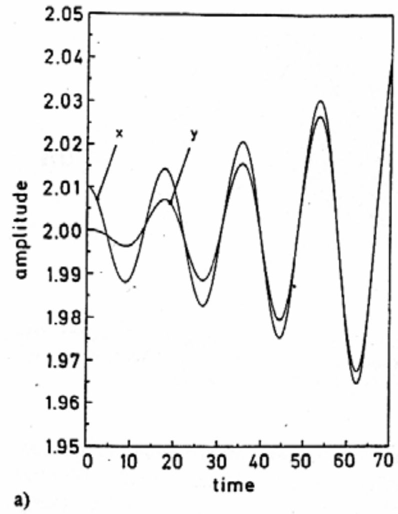
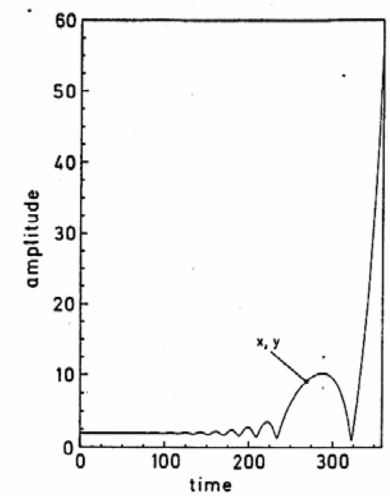


Fig. 1. Damped oscillation according to (6) and (7) in a double-vale potential resonantly drive. Plotted is the time dependence of the model x and the experimental system y. a) The solution $y = x$ is stable. The minor disturbance at $t = 0$ decreases. – b) Unlimited amplitudes can be reached. c) In this case the initial condition is $y(t = 0) = -4$, $x(t = 0) = 4$. The major disturbance remains.



a)



b)

Fig. 2. Damped oscillation in an effective Kepler potential

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 y + 0.002\,\frac{\mathrm{d}}{\mathrm{d}t}\,y + \frac{1}{y^2} - \frac{2}{y^3} = 0 \quad \text{resonantly driven.}$$

a) The solution $y = x$ is stable. The minor disturbance at $t = 0$ decreases. – b) The system can be ionised.

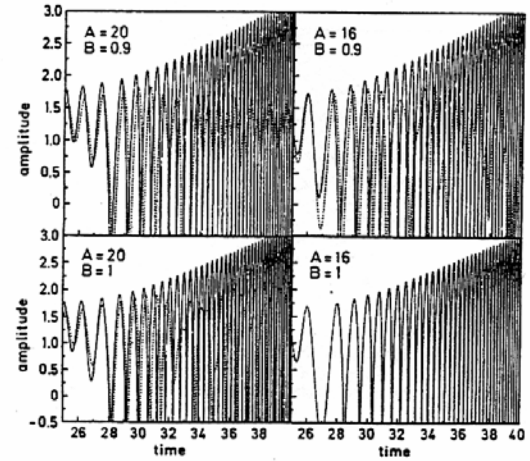


Fig. 3. The dynamics of the experiment (6) when the model (7) is varied systematically. ——— $\log x$, – – – – $\log y$. – If model (parameter A, B) and experiment ($A = 16$, $B = 1$) are equivalent, large amplitudes result. – x and y are in phase when the experiment is stimulated resonantly. This implies a phase difference of 90 degrees between perturbation and oscillator y and a phase difference of 0 degrees between the perturbation and the velocity of the oscillator.

obeys for example the differential equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 y + 0.3\,\frac{\mathrm{d}}{\mathrm{d}t}\,y - 16\,y + y^3 = 0 \qquad (6)$$

and if the model

$$\left(\frac{\mathrm{d}}{\mathrm{d}t^*}\right)^2 x + 0.3\,\frac{\mathrm{d}}{\mathrm{d}t^*}\,X - A\,x + B\,x^3 = 0 \qquad (7)$$

is used, then the unknown real parameters A and B of the model can be determined by observing the strongest reaction to stimulation. The parameters A and B are varied systematically. The corresponding driving forces are determined by the model and are used to stimulate the experimental system. The resulting amplitudes of the experimental system are plotted in Figs. 3 and 4 as functions of A and B. As can be seen from Fig. 4, the experimental system reaches the highest energy when the model and the experiment are equivalent. Figure 5 shows how much energy is absorbed by the experimental oscillator during the stimulation. Every experimental system has a limitation for the maximal

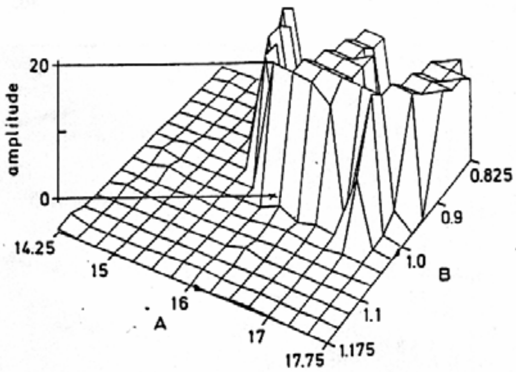


Fig. 4. The normalized energy of the experimental system (at point T, where the perturbation reaches 20) as a function of the parameters A and B of model (7).

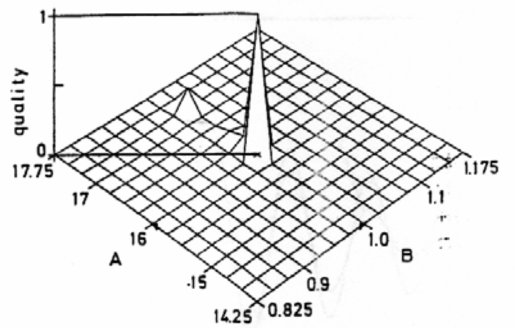


Fig. 6. The normalized quality of absorption in a measurement up to point T as a function of the parameters A and B in the non-linear system for (6) and (7).

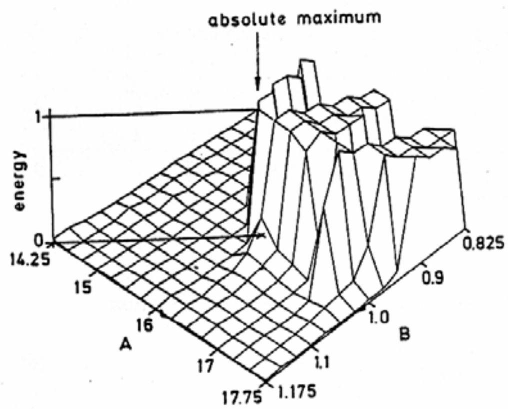


Fig. 5. The normalized absorption of energy as a function of the parameters A and B of model (7).

driving amplitude. The measurement was taken when $x(t^*(t)) \doteq 20$, i.e. the driving amplitude reaches an experimental limit. In all cases the initial value of the driving system $x(t^* = T)$ and the experiment $y(0)$ were close to the stationary state. This means that the time $t = 0$ the system can be approximated by a harmonic one.

Although the experimental system can also be stimulated to relatively large ocillations when the differential equation of the model and the experimental system do not exactly coincide, we observe a characteristic difference. At exact coincidence the driving force and the velocity of the experimental system are always in phase, otherwise not (Figure 3). In the case of coincidence this has the effect that the experimental system continuously absorbs energy from the perturbation, else energy is exchanged between experimental system and perturbation.

With the reaction power

$$B(T) = \frac{1}{T}\int_0^T \frac{\mathrm{d}E}{\mathrm{d}t}\,\theta\left(-\frac{\mathrm{d}E}{\mathrm{d}t}\right)\mathrm{d}t, \tag{8}$$

$$\theta(x) = \begin{cases} 0 & \text{for } x \leqq 0, \\ 1 & \text{for } x > 0 \end{cases}$$

this exchange of energy can be calculated. A sharp resonance line (Fig. 6) is provided by the quality factor of absorption

$$Q(T) = \frac{E(T)}{TB(T)}. \tag{9}$$

If there is a loss of energy at the transfer from the drive to the experimental system (ohmic resistance of electric conductors), the additional dissipation produced by a large exchange of energy, the quality factor $Q(T)$ should be at least indirectly measurable.

4. Summary

Resonant stimulations are possible with damped nonlinear oscillators. If a mathematical model of the dynamics of the experimental system exists which describes roughly the dynamics of the experimental system, this model can be optimized with the help of resonance spectroscopy. Besides, the standard resonance spectroscopy with sinusoidal stimulation is only then successful, when the indirect model hypothesis that the experimental system is a harmonic one, is roughly true and when also the resonance frequency is approximately known. Since the oscillation amplitude and the necessary driving forces increase strongly in time (with the method shown above) pulsed measurements are recommended for experimental investigations.

On the average, the neighbouring trajectories converge in a geometrical reconstruction of the state space of $x(t^*)$, provided the state space contracts (due to damping). Time reflection leads to an expansion of the state space. Therefore the algorithm for calculating the driving force is sensitive to the initial conditions. The Fourier spectrum of the driving mechanism is as broad and as continuous as the spectra of the damped non-linear oscillators. This non-periodical driving force can nevertheless be numerically calculated without any problems and used for the stimulation of an experimental system with an appropriate digital-analog-converter.

The shown method of resonance spectroscopy is superior to the passive methods mentioned in the beginning, especially when the experimental system is a set of identical, weakly coupled oscillators which oscillate mainly incoherently. If one gets without perturbation a compound signal of all oscillators, damped by destructive interference, a strong response emerges in resonance spectroscopy, because every single oscillator is forced to coherent oscillations due to the perturbation.

We wish to thank H. Haken, W. Kroy, H. Altmann, O. Wohofsky, P. Deisz, and Ch. Berding for many discussions. For financial support we give our thanks to Messerschmitt-Bölkow-Blohm.

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[12] The numeric integrations have been done with the program DVERK of the IMSL library (IMSL, 7500 Bellaire Boulevard, Houston, Texas, USA) a Runge-Kutta algorithm of 5th–6th order withe 500 steps per oscillation and tolerances 10^{-10}.