Resonance Spectroscopy with Chaotic Forcing Functions

Glenn Foster, Alfred W. Hübler, and Karin Dahmen

*Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801**

(Dated: June 3, 2006)

Abstract

We study resonance curves of nonlinear dynamical systems with chaotic forcing functions. We use the calculus of variations to determine the forcing function that induces the largest response. We find that the product of resonant forcing and the displacement of nearby trajectories is a conserved quantity, i.e. when the displacement dynamics is irregular, the resonant forcing function is irregular too. We compute the resonant forcing for a set of model systems and determine the response of the dynamical system to each forcing function. We show that the response is largest if the model system matches the dynamical system. We find that the signal to noise ratio is particularly large if one of the Lyapunov exponents is large.

PACS numbers: 05.45.-a, 02.30.Xx, 45.10.Hj

*Electronic address: `a-hubler@uiuc.edu`

Nonlinear dynamical systems with sinusoidal forcing functions have a large range of important applications, including synchronization [1, 2], stochastic resonance [3], and nonlinear transport phenomena [4]. Less studied is system identification with resonance curves of nonlinear systems [5]. An area that has received much less attention are resonance phenomena of nonlinear systems due to aperiodic and chaotic forcing functions. Plapp [6] and others [7] have shown, that special aperiodic driving forces can achieve a large energy transfer in damped nonlinear oscillators. Such non-sinusoidal resonant forcing functions yield a large signal-to-noise ratio. A large signal-to-noise ratio can be used for high-resolution system identification[8].

In this paper, we study resonance curves of chaotic systems with a particularly large signal-to-noise ratio. We use aperiodic forcing functions which produce a particularly large response. We compute resonance curves for chaotic Henon map dynamics, chaotic logistic map dynamics, and coupled Bernoulli maps. The Henon map is a time discrete version of the Lorenz attractor. In 1963, Edward Lorenz derived this dynamical system from the simplified equations of convection rolls arising in the equations of the atmosphere. The logistic map was introduced as a demographic model by Pierre Francois Verhulst and as a model for limited growth in biological systems by Robert May[9]. Grassberger [10] studied the chaotic dynamics in detail. The Bernoulli map is closely related to the tent map, which is a piecewise linear approximation of the logistic map.

We consider the iterated map dynamics where the mapping function f contains an unknown parameter a:

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, a) + \mathbf{F}_n + \mathbf{r}_n \tag{1}$$

where $\mathbf{x}_n \in \mathbb{R}^d$ denotes the state of the d-dimensional system at time step $n = 0, 1, \cdots, N-1$. $\mathbf{F}_n \in \mathbb{R}^d$ is a small forcing function at time step n. $\mathbf{r}_n = (r_{n,1}, r_{n,2}, \cdots, r_{n,d})$ is very small, additive, band-limited, white noise where each component at each time step is a random number with variance $< (r_{n,i})^2 >= r^2/(Nd)$ without correlations, i.e. $< r_{n,i} r_{n,j} >= 0$ for $i \neq j$ where $i, j = 1, 2, \ldots, , d$. The response is predicted with a model:

$$\mathbf{X}_{n+1} = \mathbf{f}(\mathbf{X}_n, a_m) + \mathbf{F}_n \tag{2}$$

where $\mathbf{X}_n$ is the state of the model system at time n. The model parameter a_m is within a given range $a_{min} \leq a_m \leq a_{max}$ which is assumed to contain the correct value, i.e. $a_{min} \leq a \leq a_{max}$. If the model is exact ($a = a_m$, $\mathbf{x}_0 = \mathbf{X}_0$, $\mathbf{r}_n = 0$), the difference between the observed response, $R^2(\mathbf{F}) = |\mathbf{x}_N - \mathbf{y}_N|^2$, and the predicted response $R_m^2 = |\mathbf{X}_N(a_m) - \mathbf{Y}_N(a_m)|^2$ is zero, where y_n is the unperturbed system dynamics, Y_n is the unperturbed model dynamics, and $\mathbf{F} = \{\mathbf{F_0}, \mathbf{F_1}, \cdots, \mathbf{F_{N-1}}\}$ is a forcing function. The fact that $D^2(\mathbf{F}, a_m) = R^2 - R_m^2$ is zero if the model is correct, can be used for system identification. System identification is unique if $D^2(\mathbf{F}, a_m)$ has only one root. The number of roots depends on the forcing function and the noise. In order to achieve a large signal to noise ratio we study the final response of the system to a set of forcing functions, which contains the resonant forcing function. The resonant forcing function is the forcing function which produces the largest response among all forcing functions with the same magnitude $F^2 = \sum_{n=0}^{N-1} (\mathbf{F}_n)^2$. We consider a set of forcing functions S, where all forcing functions have the same magnitude F^2, and each forcing function maximizes the final response of a particular model with model parameter a_m. Then the forcing function depends on the model parameter $\mathbf{F} = \mathbf{F}(\mathbf{a_m})$. If the model is correct the forcing function maximizes the response of the dynamical system among all forcing functions with the same magnitude. Consequently $\mathbf{F}(a)$ maximizes the response of the system among all forcing functions within the set S. The function $R(\mathbf{F}(a_m))$ is called a resonance curve. The resonance curve has an absolute maximum for $a_m = a$, i.e. $R(\mathbf{F}(a)) \geq R(\mathbf{F}(a_m))$ for all a_m in the range $a_{min} \leq a_m \leq a_{max}$.

We use the calculus of variations with Lagrange function $L = R_m^2/2 + \sum_{n=0}^{N-1} \mathbf{k}_n \left(\mathbf{x}_{n+1} - \mathbf{f}\left(\mathbf{x}_n\right) - \mathbf{F}_n\right) + \mu \left(\mathbf{F}_n\right)^2/2$ to determine the forcing function with yields the largest final response R_m^2 where $\mathbf{Y}_0 = \mathbf{X}_0$. $\mathbf{k}^{(n)}$ and μ are Lagrange multipliers. The stationary points of the Lagrange function provide necessary conditions for the maximum response. Elimination of the Lagrange multipliers $\mathbf{k}^{(n)}$ gives the following set of equations for the resonant forcing function

$$\left(J_{n+1}(b)\right)^T \mathbf{F}_{n+1} = \mathbf{F}_n \tag{3}$$

and

$$\mathbf{x}_N - \mathbf{y}_N = -\mu \mathbf{F}_{N-1} \tag{4}$$

where n=0,1,...,N-1, and $J_n(b) = (\partial f_i(b)/\partial X_j)|_{\mathbf{X}_n}$ is the Jacobi matrix evaluated at $\mathbf{X}_n$. The dynamics of a small displacement $\mathbf{d}_n = \mathbf{X}_n - \tilde{\mathbf{X}}_n$ between two neighboring trajectories at $\mathbf{X}_n$ and $\tilde{\mathbf{X}}_n$ is $\mathbf{d}^{(n+1)} = J^{(n)}\mathbf{d}^{(n)}$. Hence the scalar product of the resonant forcing and the displacement is a conserved quantity

$$P = \mathbf{F}_n \cdot \mathbf{d}_{n+1} \tag{5}$$

for $n = 0, 1, \cdots, N-1$. The resonant forcing function complements the displacement dynamics of the model. If the system is one-dimensional, then the resonant forcing is proportional to the inverse of the displacement at each time step, i.e. $F_n = P/d_{n+1}$. Thus if the displacement dynamics is periodic, then the resonant forcing has the same periodicity and if the displacement dynamics is chaotic, then the resonant forcing has the same type of aperiodicity. Figure 1 shows that the displacement and the optimal forcing function are complementary for a chaotic logistic map dynamics, $x_{n+1} = ax_n(1-x_n)+F_n$, where $a = 3.61$, $N = 15$, $F = 0.0001$ and $x_0 = 0.897$.

Next we assume that the forcing function is small and expand the Jacobi matrix about the unperturbed dynamics to lowest order, i.e. $J_n \approx (\partial f_i/\partial x_j)|_{\mathbf{y}_n}$. To lowest order, the difference between the trajectory of the driven system and the unperturbed system reads: $\mathbf{x}_N - \mathbf{y}_N = \mathbf{F}_{N-1} + \sum_{n=1}^{N-1} \left(\prod_{i=1}^{n} J_{N-i}\right) \mathbf{F}_{N-1-n}$. With Eq. (3) and Eq. (4) we obtain

$$M\mathbf{F}_{N-1} = -\mu \mathbf{F}_{N-1} \tag{6}$$

where $M = I + \sum_{n=1}^{N-1} M_n$ and $M_n = \left(\prod_{i=1}^{n} J_{N-i}\right)\left(\prod_{i=1}^{n} J_{N-i}\right)^T$. I is the identity matrix. M is a symmetric matrix with up to d orthogonal eigenvectors $\mathbf{e}_i$, where $M\mathbf{e}_i = \mu_i \mathbf{e}_i$, $i = 1, 2, \ldots, d$ and $\mathbf{e}_i^2 = 1$. The corresponding eigenvalues μ_i are positive. The eigenvectors of matrix M are the solutions of Eq. (6) $\mathbf{F}_{N-1} = \pm F_{N-1}\mathbf{e}_i$, where $F_{N-1} = |\mathbf{F}_{N-1}|$ and $\mu = -\mu_i$. Eq. (3) and Eq. (6) yield $F^2 = \mu_i F_{N-1}^2$ and $R^2 = (\mathbf{x}_N - \mathbf{y}_N)^2 = \mu_i^2 \left(F_{N-1}\right)^2 = \mu_i F^2$. Hence the final forcing which parallels the eigenvector with the largest eigenvalue of M, $\hat{\mu} = \max\{\mu_i\}$ produces the largest response, and the largest response is

$$R^2 = \hat{\mu} F^2 \tag{7}$$

and with Eq. (6) we obtain

$$\hat{\mathbf{F}}_{N-1} = \pm \frac{F}{\sqrt{\hat{\mu}}} \hat{\mathbf{e}} \tag{8}$$

where $\hat{\mathbf{e}}$ is the eigenvector that corresponds to the largest eigenvalue of M, and for $n = 0, 1, \cdots, N-2$ the time dependence of the resonant forcing function is $\mathbf{F}_n = \pm\frac{F}{\sqrt{\hat{\mu}}}\left(\prod_{i=1}^{N-1-n} \left(J_{n+i}\right)^T\right)\hat{\mathbf{e}}$.

Figure 2 shows resonance curves of a chaotic Henon map dynamics as a function of the map parameter a_m and b_m. The Henon map is $x_{1,n+1} = 1 - a(x_{1,n})^2 + x_{2,n} + F_{1,n}$ and $x_{2,n+1} = bx_{1,n} + F_{2,n}$, where $\mathbf{x}_n = (x_{1,n}, x_{2,n})$ is the state and $\mathbf{F}_n = (F_{1,n}, F_{2,n})$ is the forcing function. a and b are parameters. The magnitude of the forcing function is $F = 0.0001$, the noise level is $r = 0$, and the number of time steps is $N = 3$. The numerical values of the peak location of the resonance curve is in good agreement with the system parameters $a = 1.08$, $b = 0.3$.

For systems with only one variable $x_{n+1} = f(x_n, a) + F_n + r_n$, the eigenvalue of M is (see Eq. (6))

$$\hat{\mu} = 1 + \sum_{n=1}^{N-2} \prod_{i=1}^{n} \left(\left.\frac{\partial f}{\partial x}\right|_{y^{(N-i)}} \right)^2 \tag{9}$$

From Eq. (3) we obtain for the resonant forcing function

$$F^{(n)} = \prod_{i=1}^{N-n-1} \left.\frac{\partial f}{\partial x}\right|_{y^{(n+i)}} F^{(N-1)} \tag{10}$$

where $F^{(N-1)} = \pm F/\sqrt{\hat{\mu}}$.

The response to the resonant forcing function is $R = \sqrt{\hat{\mu}}F$. Fig. 3 shows the resonant forcing function (Eq. (10)) and the displacement dynamics for a chaotic logistic map dynamics $x^{(n+1)} = 3.61x^{(n)}(1 - x^{(n)}) + F^{(n)}$, for $n = 0, 1, \cdots, 14$, with the initial condition $y^{(0)} = x^{(0)} = 0.34$. The magnitude of the forcing function $F = 0.0001$. With Eq. (9) we compute $\hat{\mu} = 1500$. We find that the predicted response $R = 0.00387$ is close to the numerical value $R = 0.00378$.

The matix $M^{(N)}$ describes how the magnitude of a displacement grows $|\mathbf{d}^{(N)}|^2 = \left(\mathbf{d}^{(0)}\right)^T M^{(N)}\mathbf{d}^{(0)}$. If $\mu_i^{(n)}$ are the eigenvalues of $M^{(n)}$ then the Lyapunov exponents are the limits $\lambda_i = \lim_{n\to\infty} \frac{1}{2n} \ln \mu_i^{(n)}$. The set of Lyapunov exponents will be the same for almost all starting points on an ergodic attractor.

For some chaotic systems the matrices $M^{(n)}$ have approximately the same eigenvectors and eigenvalues. For instance if the Jaccobian is constant, i.e. $J^{(n)} = J^{(0)}$ for $n = 1, 2, \cdots, N-1$, then $M^{(n)} = (J^{(0)})^n((J^{(0)})^T)^n = (J^{(0)}(J^{(0)})^T)^n = (M^{(1)})^n$ since $J^{(0)}(J^{(0)})^T = (J^{(0)})^T J^{(0)}$. This is the case for coupled Bernoulli map dynamics (see Eq. (13)). If the matrices $M^{(n)}$ have approximately the same eigenvectors and eigenvalues, then the eigenvalues obey the following relation $\mu_i^{(n)} \approx (\mu_i^{(1)})^n \approx e^{2n\lambda_i}$.

If an initial displacement is parallel to the eigenvector of M that corresponds to the largest Lyapunov exponent $\hat{\lambda} = max\{\lambda_i, i = 1, 2, \cdots, d\}$, then it has the largest growth rate, i.e. $d^{(n)} = e^{n\hat{\lambda}} d^{(0)}$. The final value of the optimal forcing function $\mathbf{F}^{(N-1)}$ is parallel to the eigenvector of M that corresponds to the largest Lyapunov exponent and earlier values obey the dynamics: $|\mathbf{F}^{(n)}|^2 = \left(\mathbf{F}^{(N-1)}\right)^T M^{(N-n)} \mathbf{F}^{(N-1)} = \mu_i^{(N-n)} (F^{(N-1)})^2 = \frac{1}{\mu_i^n} (F^{(0)})^2$ Hence the the growth rate of the magnitude of the optimal forcing function is equal to the opposite of the largest Lyapunov exponent:

$$F^{(n)} \approx e^{-\hat{\lambda} n} F^{(0)} \tag{11}$$

Since $M = I + \sum_{n=1}^{N-1} M^{(n)}$ we estimate $\hat{\mu} \approx \sum_{n=0}{}^{N-1} \hat{\mu}_1^n = \frac{1-\mu_1^N}{1-\mu_1} \approx \frac{1-e^{2\hat{\lambda}N}}{1-e^{2\hat{\lambda}}}$. Then the response can be approximated by

$$R^2 \approx \frac{1 - e^{2\hat{\lambda}N}}{1 - e^{2\hat{\lambda}}} F^2 \tag{12}$$

An example for a mapping function with a constant Jaccobian is a system of two coupled Bernoulli maps:

$$\begin{pmatrix} x_{1,n+1} \\ x_{2,n+1} \end{pmatrix} = \begin{pmatrix} mod(ax_{1,n} + kx_{2,n} + F_{1,n}) \\ mod(bx_{2,n} + kx_{1,n} + F_{2,n}) \end{pmatrix} \tag{13}$$

where the function $mod(x) = x - \lfloor x \rfloor$ returns the decimal part of x. a and b are the growth rates and k is the coupling constant. We assume that $a > b \geq 0$. For the corresponding model dynamics with the parameters a_m, b_m, and k_m, the eigenvalues of M_1 are $\hat{\mu}_1(a_m, b_m, k_m) = 0.5(a_m^2 + b_m^2 + 2k_m^2 + (a_m + b_m)\sqrt{(a_m - b_m)^2 + 4k_m^2})$ and the eigenvectors $\mathbf{e} = (a_m - b_m + \sqrt{(a_m - b_m)^2 + 4k_m^2}, 2k_m) / \sqrt{(\left(a_m - b_m \pm \sqrt{(a_m - b_m)^2 + 4k_m^2}\right)^2 + 4k_m^2)}$. Since the Jacobian $J_n(a_m, b_m, k_m)$ is symmetric and constant, the eigenvectors of the M_1 are eigenvectors of M_n, and the Lyapunov exponents are $\lambda_{1/2} = \frac{1}{2} \ln \mu_{1/2}^{(1)}$ and the largest Lyapunov exponent is $\hat{\lambda} = \frac{1}{2} \ln(\hat{\mu})$. If $\hat{\mu}(a, b, k) > 1$ the unperturbed dynamics is chaotic. Hence $\mathbf{F}^{(n)} = \pm (J)^{N-1-n} \frac{F}{\sqrt{\hat{\mu}}} \hat{\mathbf{e}}$. The peak value of the resonance curve is given by Eq. (14).

If $k = 0$ then $\hat{\mu}_J = a_1$, $\hat{\mu} = \frac{a_1^{2N}-1}{a_1^2-1}$, $\hat{\mathbf{e}} = (1, 0)$, $\mathbf{F}^{(n)} = \pm(a_1^{N-n-1} F/\sqrt{\hat{\mu}}, 0) = (F^{(0)}/a_1^n, 0)$, where $F^{(0)} = \pm a_1^{N-1} F/\sqrt{\hat{\mu}}$ and

$$R^2 = F^2 \frac{(a^N a_m^N - 1)^2 (a_m^2 - 1)}{(a_m^{2N} - 1)(a a_m - 1)^2} \tag{14}$$

The resonance curve does not depend on b_m. Hence the resonance curve can not be used to determine the parameter of the less chaotic dynamics b_m. For $k = 0$ the systems contains two decoupled Bernoulli maps, where $\lambda_i = \ln|a_i|$, $i = 1, 2$ is the Lyapunov exponent of each map. The resonant forcing function is in the direction of the map with the larger Lyapunov exponent. Hence if both maps have a positive Lyapunov exponent and therefore both are chaotic, then the resonant forcing function forces only the map which is more chaotic. There is no forcing of the less chaotic map. For a periodic forcing $F_{1,n} = (-1)^n F/\sqrt{N}$ and $F_{2,n} = 0$ the difference between the system response R^2 and the model response R_m^2 is

$$D^2 = \left(\left(\frac{1 - (-a)^N}{a + 1} \right)^2 - \left(\frac{1 - (-a_m)^N}{a_m + 1} \right)^2 \right) \frac{F^2}{N} \tag{15}$$

Figure 4 shows the numerical and theoretical resonance curve for a Bernoulli map dynamics and the deviation D^2. The response to the sinusoidal forcing function and D are much smaller than the response to the optimal forcing function.

Finally we compute the response to random forcing $\mathbf{r}_n = (r_{1,n}, r_{2,n}, \cdots, r_{d,n})$ where each component of the forcing function at each time step is a random number with variance $< (r_{i,n})^2 >= F^2/(Nd)$ and without correlations $< r_{i,n} r_{j,n} >= 0$ for $i \neq j$. Then the expectation value of the response is

$$R_r^2 = \left(\frac{1}{d} \sum_{i=1}^{d} \frac{1 - e^{2\lambda_i N}}{1 - e^{2\lambda_i}} \right) \frac{r^2}{N} \tag{16}$$

From Eq. (12) and Eq. (16) we conclude that the response for the optimal forcing is large compared to response from random forcing, if the largest Lyapunov exponent is much larger than the other Lyapunov exponents. Fig. 5 shows the signal-to-noise ratio, i.e. the ratio response for optimal forcing and random forcing, R^2/R_r^2, as a function of the largest Lyapunov exponent for a chaotic Bernoulli map dynamics. The signal-to-noise ratio is particularly large, if the largest Lyapunov exponent is much larger than the other Lyapunov exponents.

In summary, we compute resonances curves of nonlinear dynamical systems with chaotic forcing functions (see Fig. 2, Fig. 3, and Fig. 4). We use the calculus of variations to determine the forcing function that induces the largest response (Eq. (3) and Eq. (4)). We find that the product of resonant forcing and the displacement of nearby trajectories is a conserved quantity (Eq. (5)), i.e. when the displacement dynamics is irregular, the resonant forcing function is irregular too (see Fig. 1). Figure 2 illustrates that the response is largest if the model system matches the dynamical system. Figure 5 shows that the signal to noise ratio is particularly large if one of the Lyapunov exponents is large.

Acknowledgments

This material is based upon work supported by the National Science Foundation Grant No. NSF PHY 01-40179 and NSF DMS 03-25939 ITR. A.H. thanks the Santa Fe Institute for support.

---

[1] T. Eisenhammer, A. Hubler, T. Geisel, and E. Luscher, Phys. Rev. A **41**, 3332 (1990).

[2] I. Siddiqi, R. Vijay, F. Pierre, C. M. Wilson, L. Frunzio, M. Metcalfe, C. Rigettiand, R. J. Schoelkopf, M. H. Devoret, D. Vion, et al., Phys. Rev. Lett. **94**, 027005 (2005).

[3] R. L. Badzey and P. Mohanty, Nature **437**, 962 (2005).

[4] S. Wimberger, R. Mannella, O. Morsch, and E. Arimondo, Phys Rev Lett. **94**, 130404 (2005).

[5] S. Krempl, T. Eisenhammer, A. Hubler, G. Mayer-Kress, and P. W. Milonni, Phys. Rev. Lett. **69**, 430 (1992).

[6] B. B. Plapp and A. W. Hubler, Phys. Rev. Lett. **65**, 2302 (1990).

[7] C. Wargitsch and A. Hubler, Phys. Rev. E **51**, 1508 (1995).

[8] K. Chang, A. Kodogeorgiou, A. W. Hubler, and E. A. Jackson, Physicsa D **51**, 99 (1991).

[9] R. M. May, Nature **261**, 459 (1976).

[10] P. Grassberger, Journal of Statistical Physics **26**, 173 (1981).

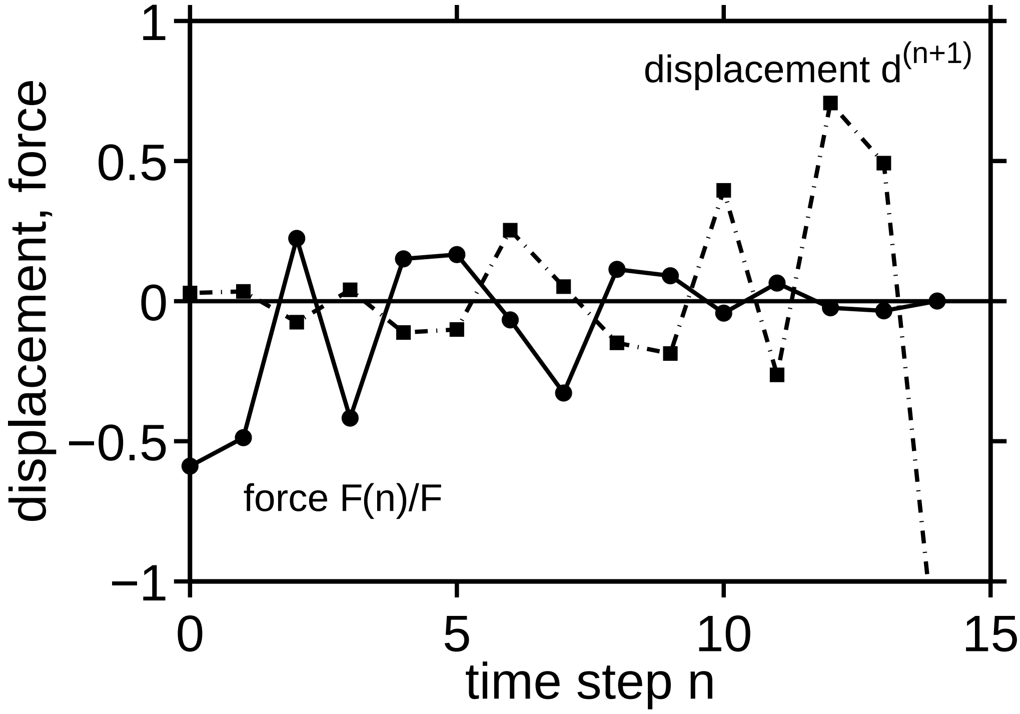

FIG. 1: The resonant forcing $F^{(n)}$ (circles) and the displacement of two neighboring trajectories $d^{(n+1)}$ (squares) versus time step n for a chaotic logistic map dynamics. This plot illustrates that the resonant forcing complements the displacement of neighboring trajectories of the unperturbed system, i.e. $F^{(n)}d^{(n+1)} = constant$. When the magnitude of the displacement is large, then the magnitude of the resonant force is small, and if the displacement is positive, the resonant force is negative.

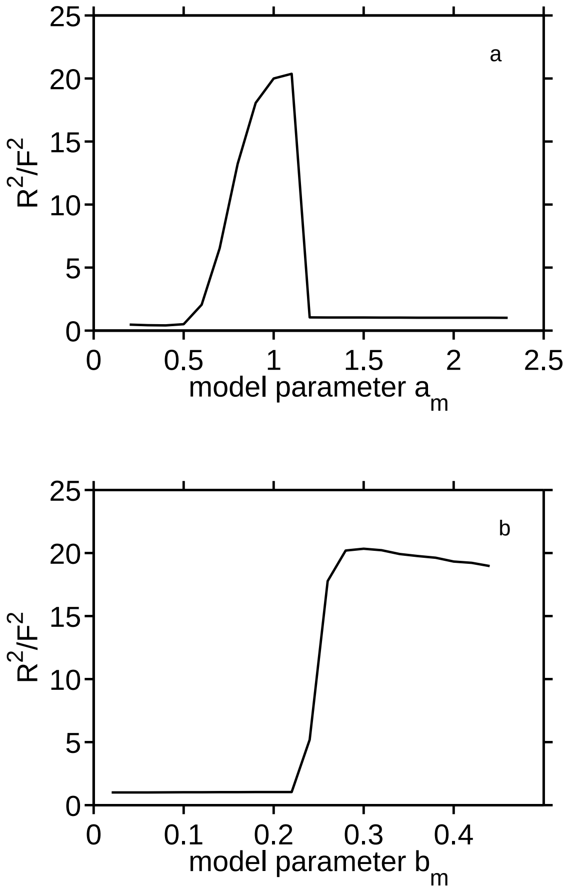

FIG. 2: The resonance curve of a chaotic Henon map versus model parameter a_m, where $b_m = b$ (a) and versus model parameter b_m where $a_m = a$. The parameters are $a = 1.08$, $b = 0.3$, $N = 3$ and $F = 0.0001$.

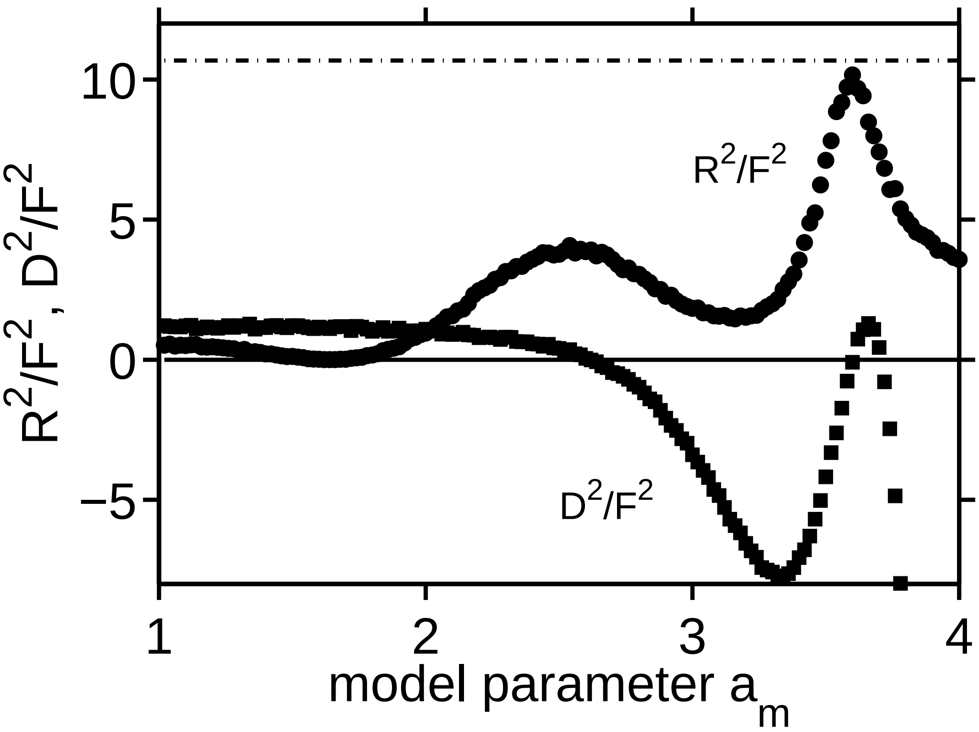

FIG. 3: The resonance curve for a chaotic logistic map (circles) and deviation D between the response model and the response of the dynamical system for a set of sinusoidal forcing functions (squares) versus the model parameter a_m. The number of time steps is $N = 4$, the noise level is $r = 0.0005$, and the magnitude of the forcing function is $F = 0.001$. The dashed line indicates the theoretical result for the maximum of the resonance curve given by Eq. (7). The maximum of the resonance curve, as well as one of the three roots of D are close to the parameter value of the dynamical system, $a = 3.6$.

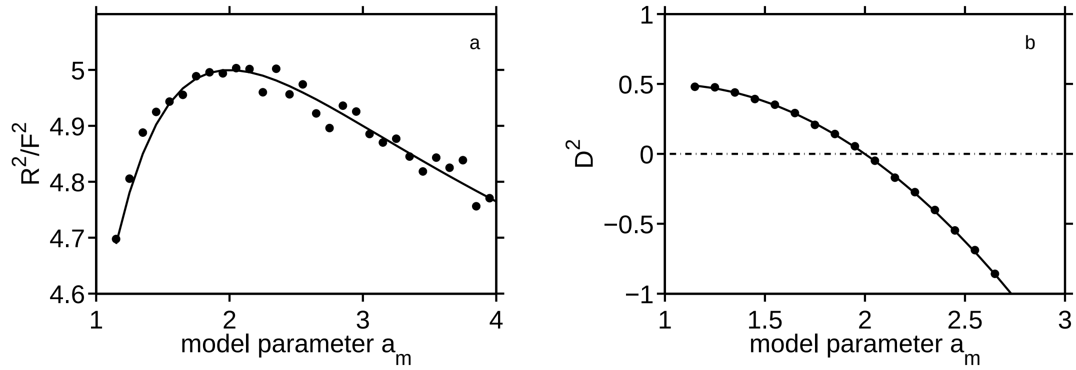

FIG. 4: The resonance curve(a) and the difference between the model response and the system response versus the model parameter (b) for a chaotic Bernoulli map with parameter $a = 2$. The continuous lines are the theoretical values given by Eq. (15) and Eq. (16). The resonance curve has an absolute maximum if the model parameter matches the system parameter. In contrast, the difference of the response is zero if the model parameter matches the system parameter.

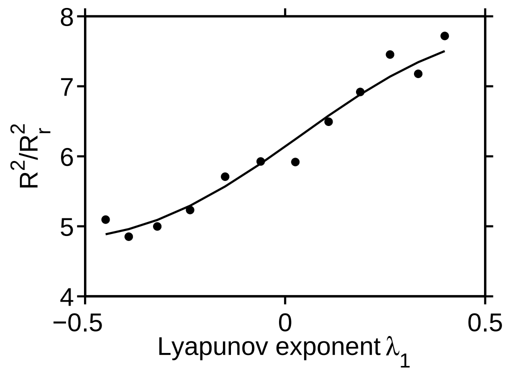

FIG. 5: Signal-to-noise ratio R^2/r_r^2 versus the largest Lyapunov exponent of a coupled Bernoulli map, where $N = 4$, $a_2 = 0.5$, $k = .2$, and $F = r = 0.0001$. The continuous line is the theoretical value given by Eq. (12) and Eq. (16). The bullets are the expectation values determined from 1000 simulations. This figure illustrates that the signal to noise ratio is particularly large if one Lyapunov exponent is significantly larger than the others.