Critical Phenomena in Gravitational Collapse: An Analytical Model

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(May 23, 2003)

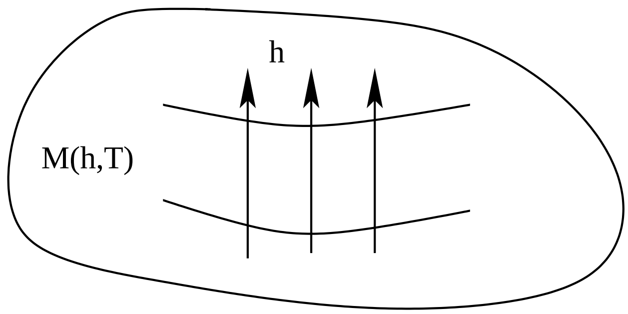
Complex Systems Symposium
UIUC, Urbana, Illinois, May 19 - 21, 2003

- Introduction to Critical Collapse.
- Critical Collapse of a Cylindrical Scalar Field: An Analytical Model.
- Current & Future Work

I. INTRODUCTION TO CRITICAL COLLAPSE

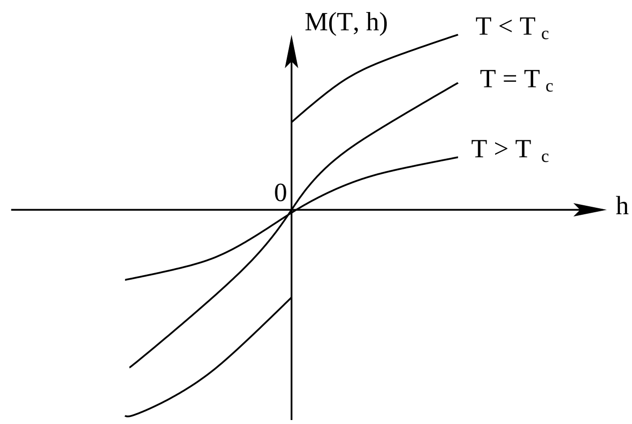
A. *Phase Transitions and Critical Phenomena in Statistical Mechanics.* For example, **Ferromagnet**:

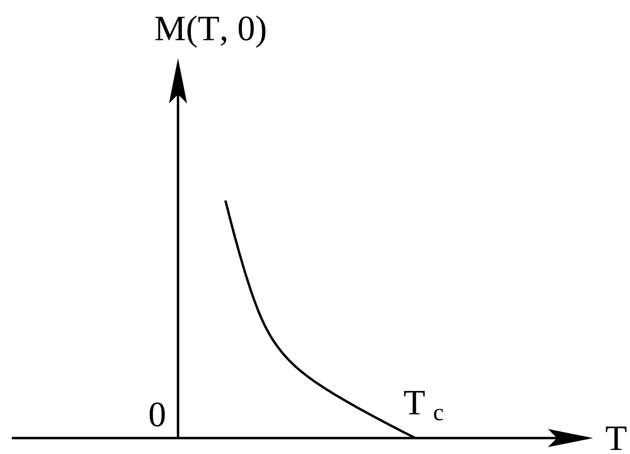
FIGURES



h: magnetic field; T: the temperature;

$M(T,h)$: magnetization.

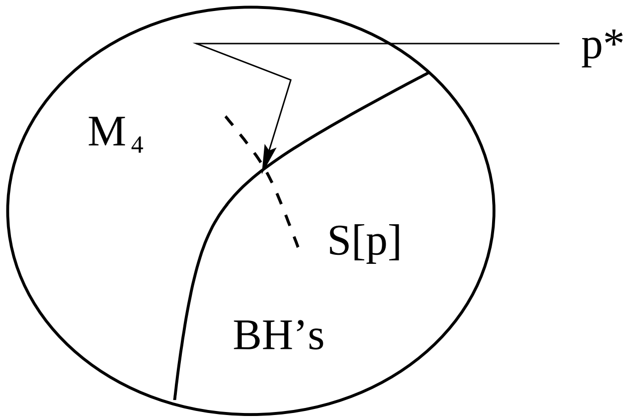




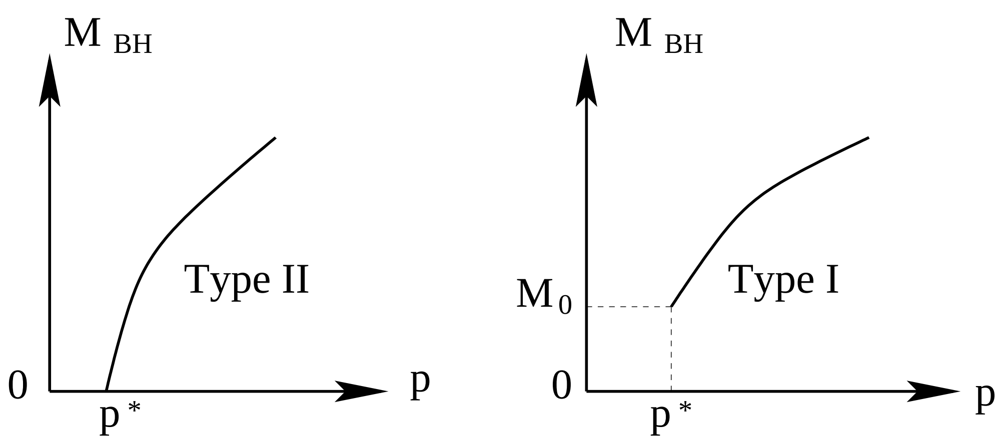
$$M(T,0) \propto (T_c - T)^\gamma, \qquad \gamma \approx 0.32.$$

B. *Critical Phenomena in Gravitational Collapse*

– **Phase Space**:



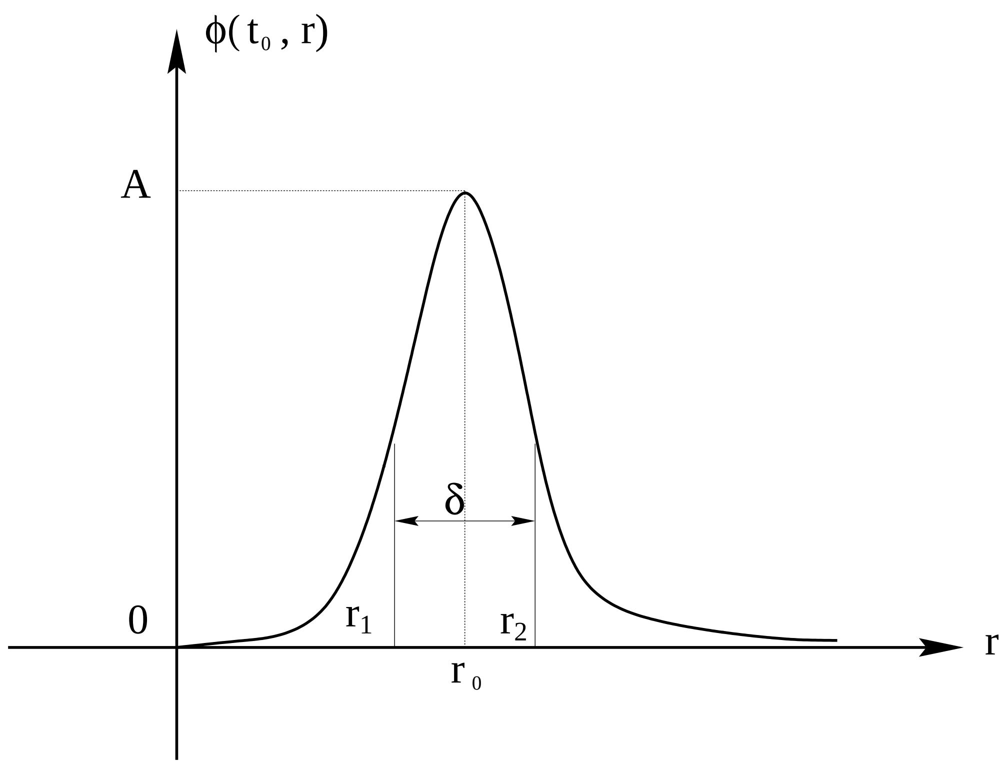
– **The Mass of Black Holes**:



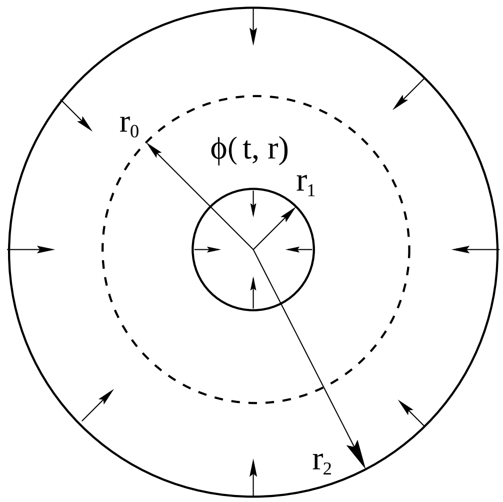
In 1993, Choptuik *numerically* studied gravitational collapse of a **scalar field** with **spherical symmetry**:

$$\phi(t_0, r) = A\left(\frac{r}{r_0}\right)^3 \exp\left\{-\left(\frac{r - r_0}{\delta}\right)^q\right\},$$

A, r_0, δ, q: arbitrary constants.



The initial configuration of the scalar field at $t = t_0$.



The parameter p can be any of the four parameters:

$$p = \{A,\ r_0,\ q,\ \delta\}.$$

For example, fixing $r_0,\ \delta$ and q, we obtain a family of initial data, $S[A]$. For this family, there exists $A = A^*$, such that

$$\begin{cases} \text{Black Holes are formed}, & A > A^*, \\ \text{Critical Solution}, & A = A^*, \\ \text{No Black Holes are formed}, & A < A^*. \end{cases} \quad (1)$$

— Four families of initial data: $S[A]$, $S[r_0]$, $S[\delta]$ and $S[q]$, and four corresponding critical solutions $S[A^*]$, $S[r_0^*]$, $S[\delta^*]$ and $S[q^*]$.

The following was found numerically:

- **All** the critical solutions are **identical** (or *Universal*).

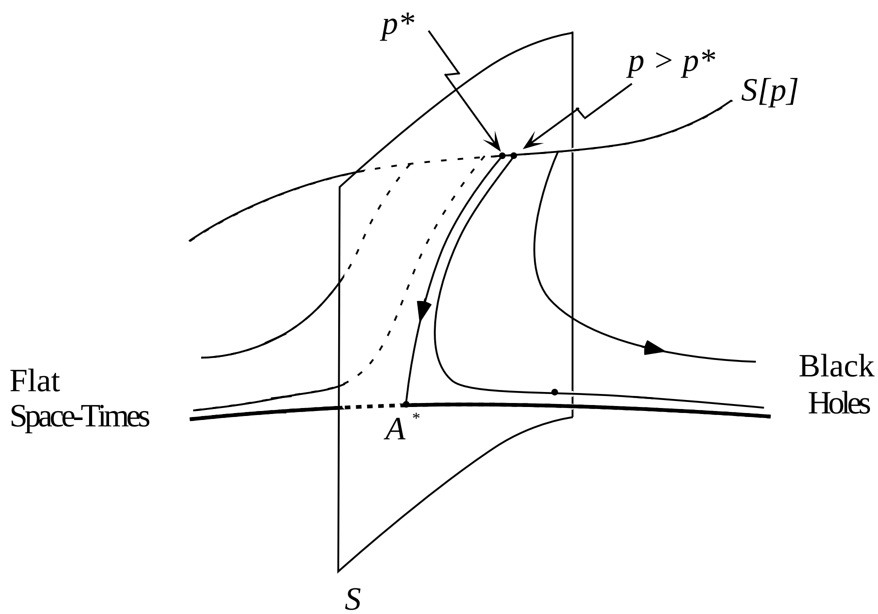
- The mass of black holes takes the **scaling form**,

$$M_{BH} \propto (p - p*)^{\gamma}, \tag{2}$$

  γ: Another **dimensionless universal** constant, $\gamma \approx 0.37$. That is, the collapse is **Type II**.

Universality of the critical solution and the exponent γ, as well as the scaling form of the black hole mass (2), all give rise to the name: *Critical Phenomena in Gravitational Collapse.*

C. *Interpretations in Terms of RG*:



– S: the critical surface of codimension one. Generic data $S[p]$ always pass S at $p = p^*$;

– All the data on S collapses to the fixed point A^*;

– All the details of initial data are soon washed out, and the collapse is very similar to the critical one.

– The self-similarity persists almost to A^*.

D. *Two Main Approaches*:

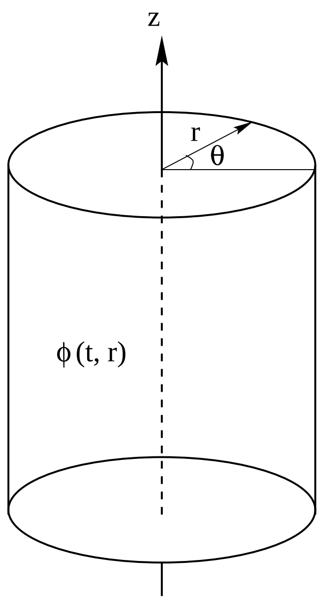
- `Numerical Simulations`.
- `Semi-analytical approach (Type II)`:

  (a) Look directly for the critical solution, by imposing certain symmetry, such as, self-similarity.

  (b) The linear perturbations. Find out the unique unstable mode, k_1, and thereby the exponent is read off

$$\gamma = \frac{1}{|k_1|}. \tag{3}$$

II. Critical Collapse of a Cylindrical Scalar Field



A. Self-similar Solutions:

$$ds^2 = e^{-M(\xi)}\left(dt^2 - dr^2\right)$$
$$-r^2 e^{-S(\xi)}\left(e^{V(\xi)}dz^2 + e^{-V(\xi)}d\theta^2\right), \qquad (4)$$

where $\xi \equiv r/(-t)$, and

$$M(\xi) = 2q^2 \ln\left(1-\xi^2\right), \quad S(\xi) = \ln(\xi),$$
$$V(\xi) = -\ln(\xi), \quad \phi(t,r) = 2q\ln(t), \qquad (5)$$

q: a free parameter.

B. Linear Perturbations:

The linear perturbations can be written as

$$F(\tau, \xi) = F_0(\xi) + \epsilon F_1(\xi) e^{k\tau}, \tag{6}$$

$F \equiv \{M, S, V, \varphi\}$;

ϵ: very small;

$F_0(\xi)$: the background self-similar solutions;

$F_1(\xi)$: the linear perturbations.

$$Re(k) = \begin{cases} \geq 0, & \text{unstable modes,} \\ < 0, & \text{stable modes.} \end{cases} \tag{7}$$

The perturbations:

$$\begin{aligned} kM_1(\xi) &= 2\xi^2 S_1'' + 2\xi V_1' + 2\xi \left[(1+k) - 4q^2 \frac{\xi^2}{1-\xi^2}\right] S_1' \\ &\quad + kV_1 + k\left(1 - 4q^2 \frac{\xi^2}{1-\xi^2}\right) S_1 + 4q\xi\varphi_1', \\ S_1(\xi) &= \frac{1}{\xi}\left[c_1(1+\xi)^{2-k} + c_2(1-\xi)^{2-k}\right], \end{aligned} \tag{8}$$

$$y(1-y)\frac{d^2Z_i}{dy^2}+[e-(a+b+1)y]\frac{dZ_i}{dy}-abZ_i$$
$$=\frac{1}{4y^{1/2}}f_i(y), \tag{9}$$

with $y \equiv \xi^2$, $\{Z_i\} = \{V_1, \varphi_1\}$, $a = b = k/2$, $e = 1$, and

$$\begin{aligned}
f_1(y) &\equiv k\xi S_1 - \left(1-\xi^2\right) S_1' \\
&= \frac{1}{y}\Big\{c_1\left[1-(2-k)y^{1/2}+y\right]\left(1+y^{1/2}\right)^{2-k} \\
&\quad +c_2\left[1+(2-k)y^{1/2}+y\right]\left(1-y^{1/2}\right)^{2-k}\Big\}, \\
f_2(y) &\equiv 2q\xi\left(\xi S_1' + kS_1\right) \\
&= -2q\Big\{c_1\left[(1-k)-y^{1/2}\right]\left(1+y^{1/2}\right)^{1-k} \\
&\quad +c_2\left[(1-k)+y^{1/2}\right]\left(1-y^{1/2}\right)^{1-k}\Big\}.
\end{aligned} \tag{10}$$

The general solutions for $V_1(\xi)$, $\varphi_1(\xi)$:

$$V_1(z) = \left(a_1^{(2)} + A_1^{(2)}(z)\right) F_1^{(1)}(z) + \left(a_1^{(1)} - A_1^{(1)}(z)\right) F_1^{(2)}(z),$$
$$\varphi_1(z) = \left(a_2^{(2)} + A_2^{(2)}(z)\right) F_1^{(1)}(z) + \left(a_2^{(1)} - A_2^{(1)}(z)\right) F_1^{(2)}(z),$$
$$(11)$$

$a_j^{(i)}$'s: integration constants, and

$$A_j^{(i)}(z) \equiv \int^z \frac{f_j(z) F_1^{(i)}(z) dz}{z\left(1 - z^2\right)\Delta(z)},$$
$$\Delta(z) \equiv F_1^{(2)}(z)\frac{d}{dz}\left(F_1^{(1)}(z)\right) - F_1^{(1)}(z)\frac{d}{dz}\left(F_1^{(2)}(z)\right),$$
$$F_1^{(1)}(z) = F\left(\frac{1}{2}k, \frac{1}{2}k; 1; z^2\right),$$
$$F_1^{(2)}(z) = F\left(\frac{1}{2}k, \frac{1}{2}k; k; 1 - z^2\right). \qquad (12)$$

— After properly imposing boundary conditions, it is found that for any given n, the solution has

$$N = n - 3, \qquad n \equiv \frac{1}{1 - 2q^2} > 1. \qquad (13)$$

unstable modes. Thus, **the solution with $n = 4$ is critical.**

III. Current & Future Work

- Effects of Angular Momentum;
- Quantum Effects;
- Observational consequences.
- Application of Renormalization Group Theory to critical collapse.

References:

- C. Gundlach, "*Critical phenomena in gravitational collapse: Living Reviews*," gr-qc/0001046 (2000).
- AW, "*Critical Phenomena in Gravitational Collapse: The Studies So Far*," gr-qc/0104073 (2001).